Modeling approaches for switching converters

by

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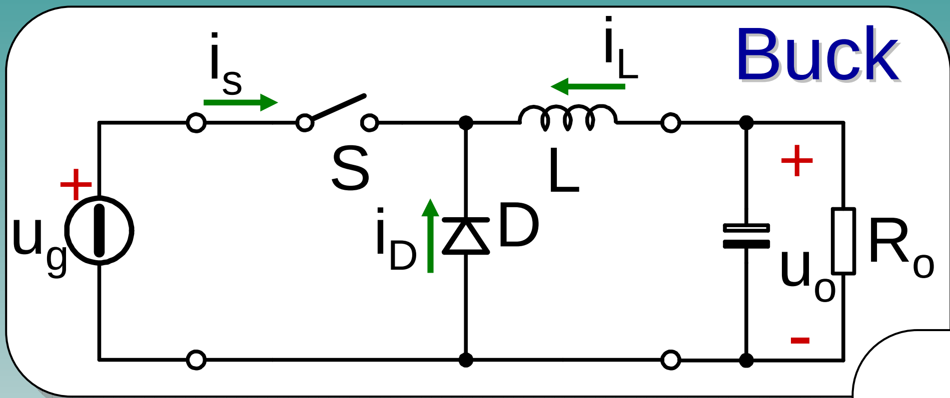
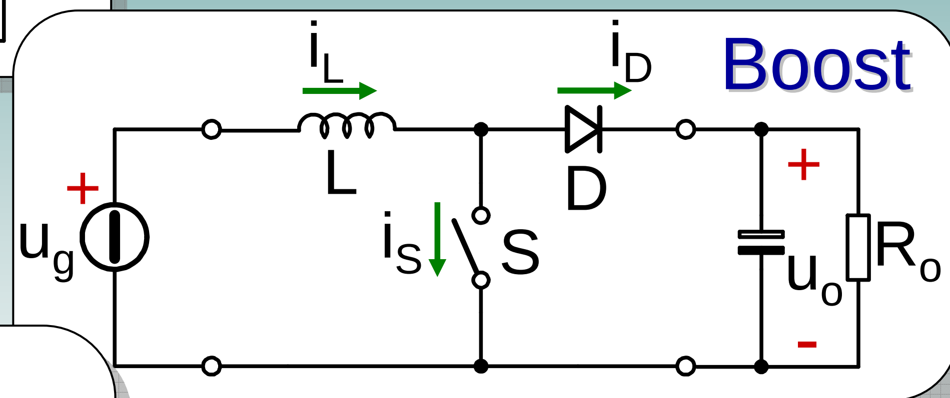
Summary of the presentation

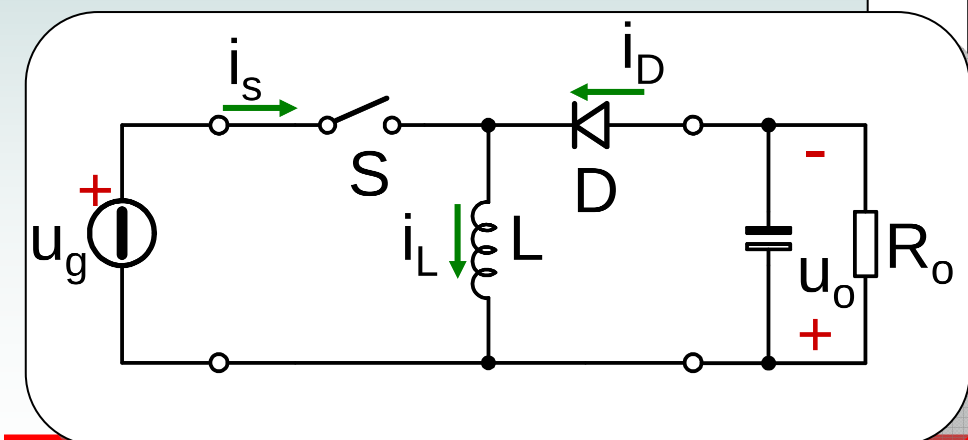
PWM converters

- Switching cell average model in continuous conduction mode (CCM)
- Switching cell average model in discontinuous conduction mode (DCM): first-order model

Basic DC-DC Converter topologies: 2° order

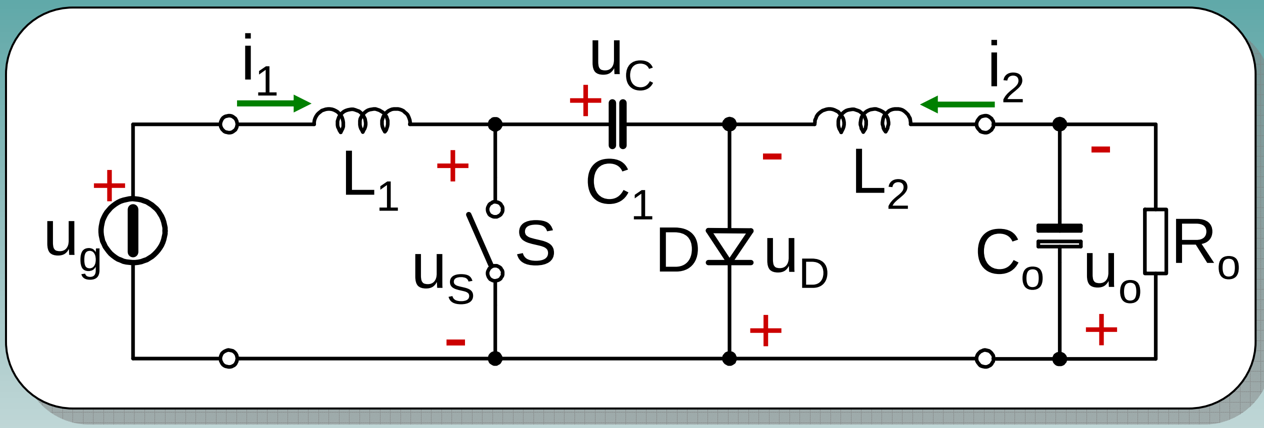






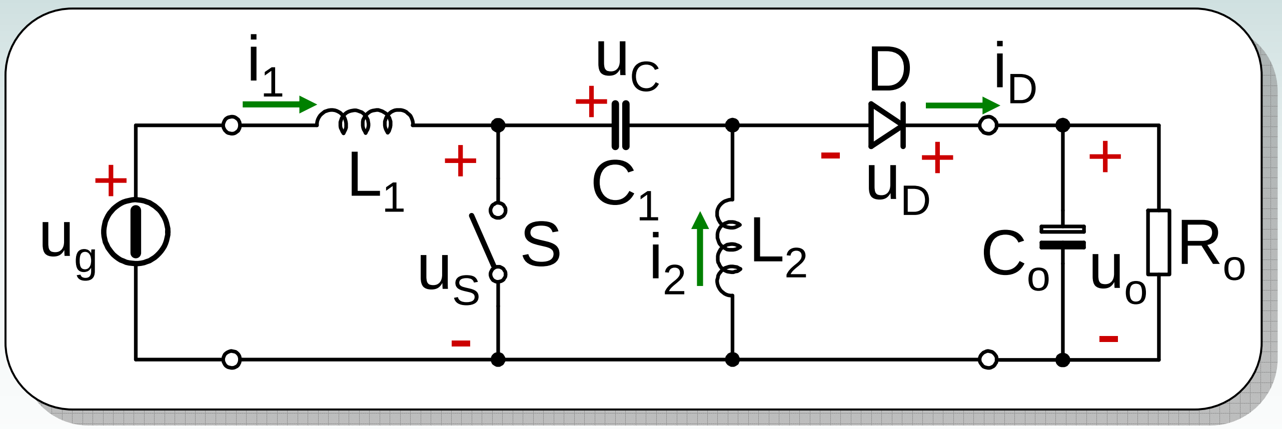
Buck-Boost

Basic DC-DC Converter topologies: 4° order



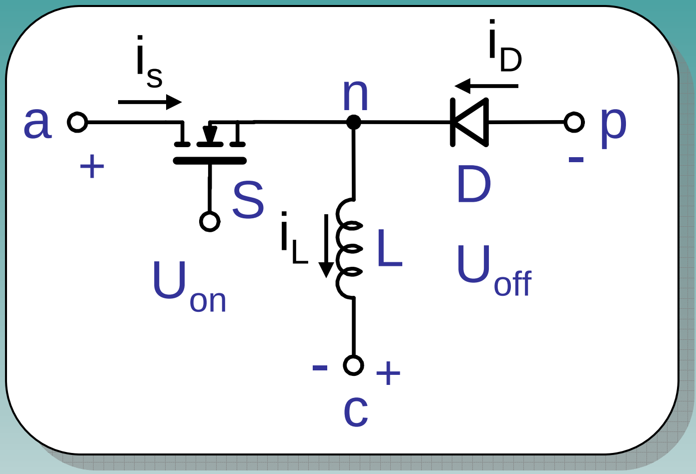
Cuk

SEPIC



Commutation Cell for 2° order converters

2° order converters can be described by a unique commutation cell:

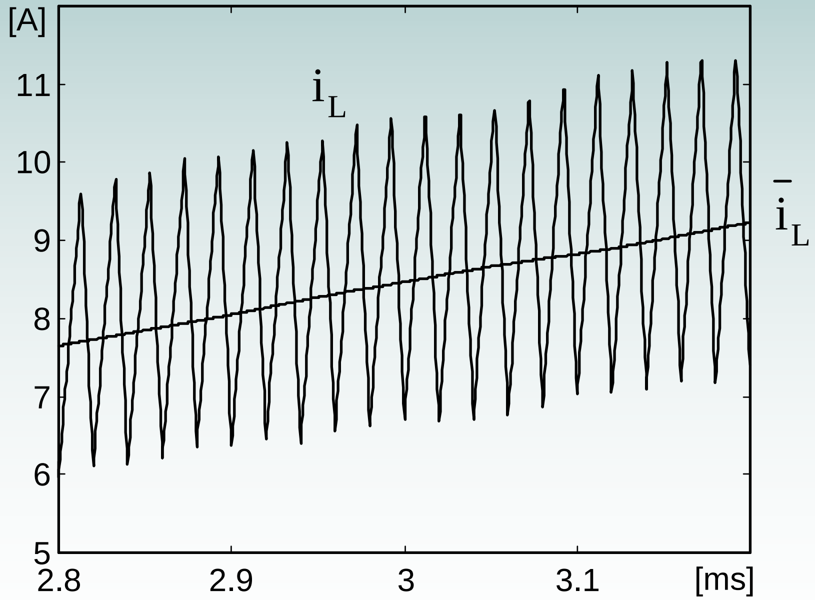


	Buck	Boost	Buck-boost
U_{on}	U_g-U_o	U_g	U_g
U_{off}	U_o	U_o-U_g	U_o
U_{on}+ U_{off}	U_g	U_o	U_g+U_o
i_g	i_S	i_L	i_S
i_o	i_L	i_D	i_D

Averaging

Moving average: $$\bar{x}(t) = \frac{1}{T_s} \int_{t-T_s}^{t} x(\tau)\, d\tau$$

Example: instantaneous and average inductor current in transient condition



Average model: CCM

- Switching frequency ripples are neglected
- Only low-frequency dynamic is investigated

Example: inductors $u_L(t) = L\frac{di_L(t)}{dt}$

$$\overline{u}_L(t) = \frac{1}{T_S}\int_{t-T_S}^{t} u_L(\tau)d\tau = \frac{L}{T_S}\int_{i_L(t-T_S)}^{i_L(t)} di_L = L\left[\frac{i_L(t) - i_L(t-T_S)}{T_S}\right]$$

Average model: CCM

$$\frac{d\bar{i}_L(t)}{dt} = \frac{d}{dt}\left[\frac{1}{T_S}\int_{t-T_S}^{t} i_L(\tau)d\tau\right] = ?$$

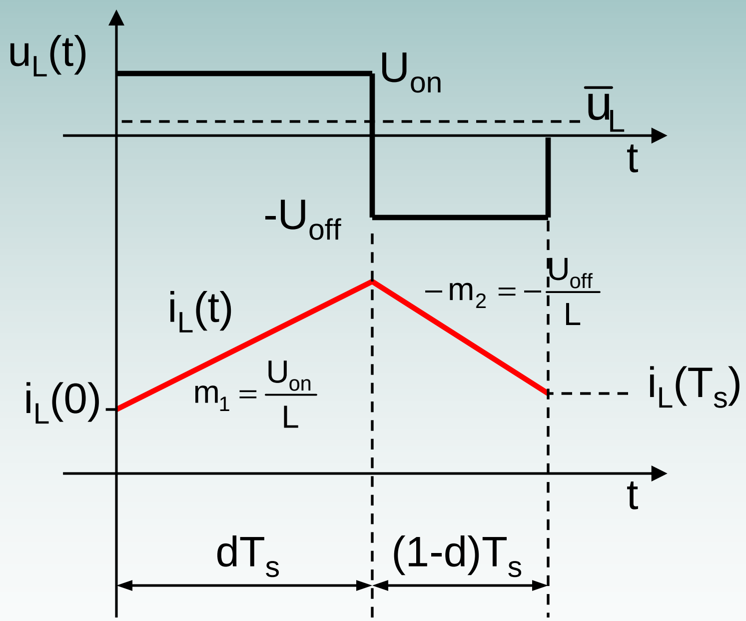
$$\phi(t) = \int_{\alpha(t)}^{\beta(t)} f(t,\tau)d\tau = \phi(t,y,z) = \int_{y}^{z} f(t,\tau)d\tau \quad \text{with } y = \alpha(t),\ z = \beta(t)$$

$$\frac{d\phi(t)}{dt} = \int_{\alpha(t)}^{\beta(t)} \frac{df(t,\tau)}{dt}d\tau - f(t,\alpha(t))\frac{d\alpha(t)}{dt} + f(t,\beta(t))\frac{d\beta(t)}{dt}$$

$$\frac{d\bar{i}_L(t)}{dt} = \frac{i_L(t) - i_L(t-T_S)}{T_S} \quad \Rightarrow \quad \bar{u}_L(t) = L\frac{d\bar{i}_L(t)}{dt}$$

Averaging approximation

Non steady-state inductor current waveform:



$$i_L(dT_s) = i_L(0) + \frac{U_{on}}{L} dT_s$$

$$i_L(T_s) = i_L(dT_s) - \frac{U_{off}}{L}(1-d)T_s$$

$$\Downarrow$$

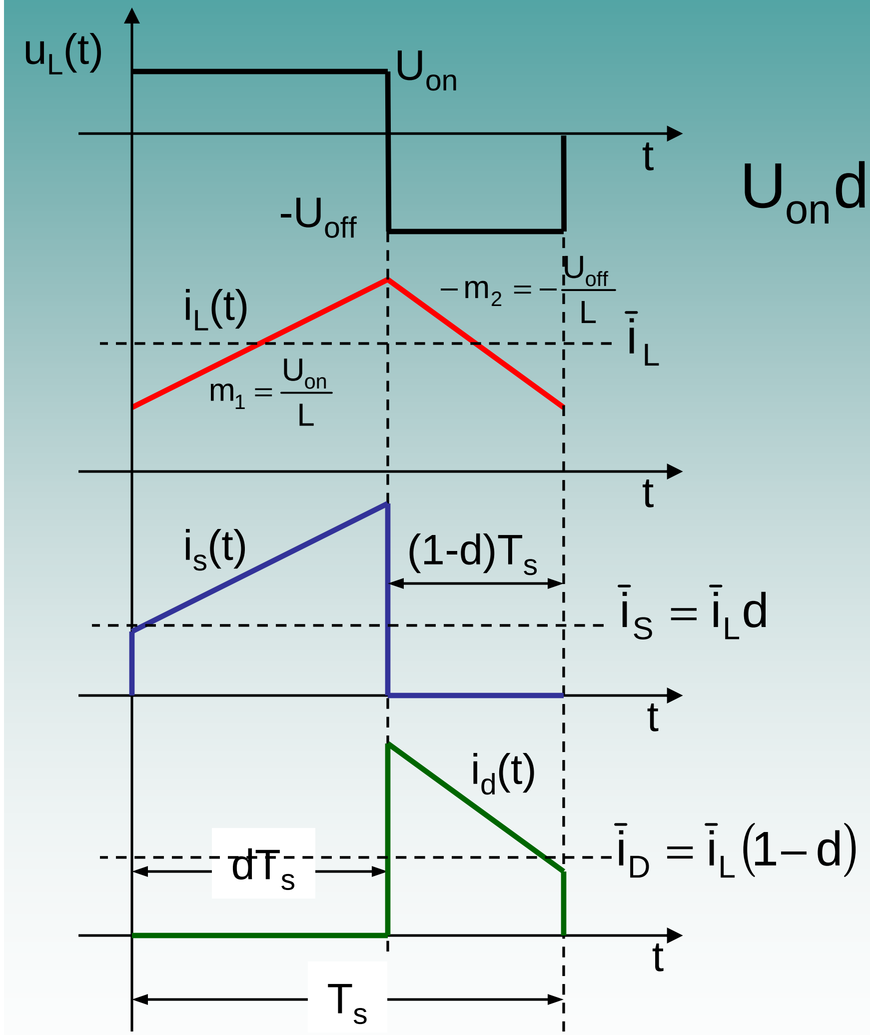
$$i_L(T_s) = i_L(0) + \frac{U_{on}}{L} dT_s - \frac{U_{off}}{L}(1-d)T_s$$

$$= i_L(0) + \frac{\overline{u}_L}{L} T_s$$

Averaging

- Reactive element voltage-current relations remain valid also for average quantities;
- for inductors, the current variation in a switching period can be calculated by integrating their average voltage;
- for capacitors, the voltage variation in a switching period can be calculated by integrating their average current.

Continuous conduction mode - CCM



At steady-state: $\bar{u}_L = 0$

$$U_{on} d T_S = U_{off}(1-d) T_S \Rightarrow \frac{U_{on}}{U_{off}} = \frac{1-d}{d}$$

Buck: $M = d$

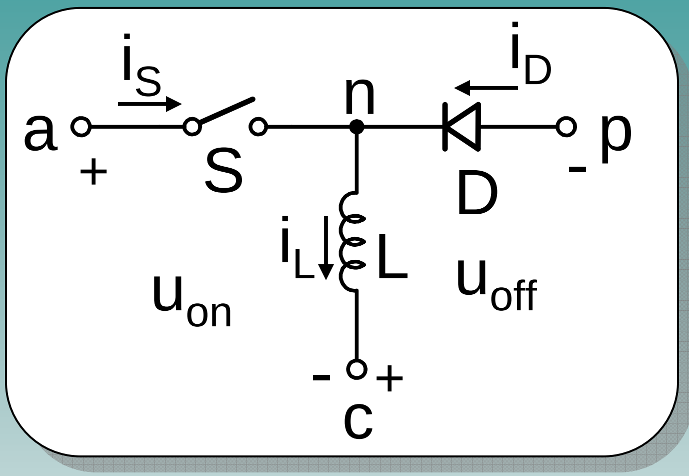
Boost: $M = \frac{1}{1-d}$

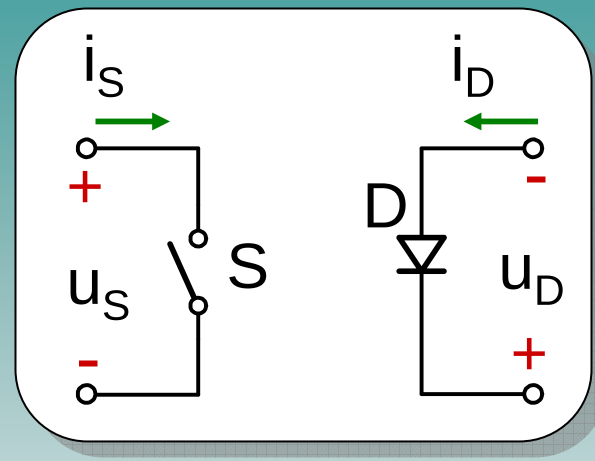
Buck-Boost: $M = \frac{d}{1-d}$

Boundary CCM-DCM:

$$\bar{i}_{L\,lim} = \frac{\Delta i_{Lpp}}{2} = \frac{U_{on}}{2Lf_S} d = \frac{U_{off}}{2Lf_S}(1-d)$$

Switching cell average model: CCM





Non linear components

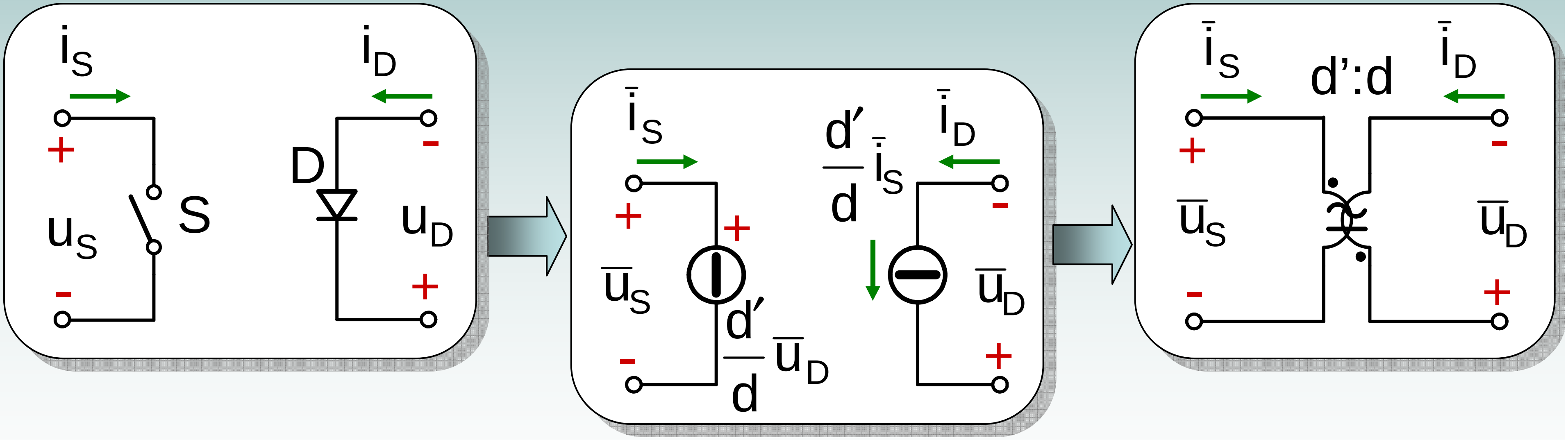
Average switch and diode voltages and currents:

$$\begin{cases} \bar{u}_S = d'\left(\bar{u}_{on} + \bar{u}_{off}\right) \\ \bar{u}_D = d\left(\bar{u}_{on} + \bar{u}_{off}\right) \end{cases} \Rightarrow \quad \bar{u}_S = \frac{d'}{d}\bar{u}_D \qquad \begin{cases} \bar{i}_S = d\,\bar{i}_L \\ \bar{i}_D = d'\,\bar{i}_L \end{cases} \Rightarrow \quad \bar{i}_S = \frac{d}{d'}\bar{i}_D$$

d'=1-d = complement of duty-cycle

Switching cell average model: CCM

$$\begin{cases} \overline{u}_S = d'(\overline{u}_{on} + \overline{u}_{off}) \\ \overline{u}_D = d(\overline{u}_{on} + \overline{u}_{off}) \end{cases} \Rightarrow \overline{u}_S = \frac{d'}{d}\overline{u}_D \qquad \begin{cases} \overline{i}_S = d\,\overline{i}_L \\ \overline{i}_D = d'\,\overline{i}_L \end{cases} \Rightarrow \overline{i}_S = \frac{d}{d'}\overline{i}_D$$

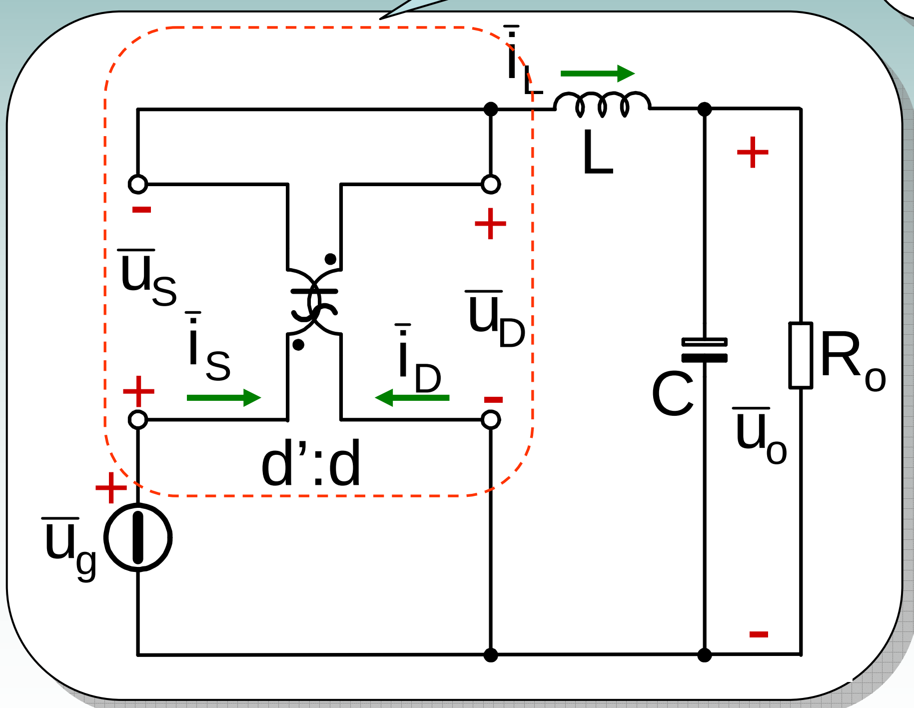


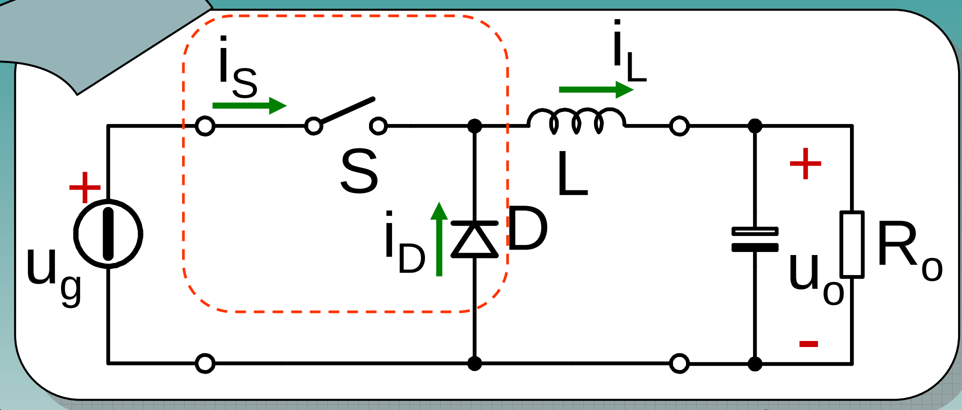
d'=1-d = complement of duty-cycle

Switching cell average model: CCM

- The non-linear components (switch and diode) are replaced by controlled voltage and current generators representing the relations between average voltage and currents;
- These controlled voltage and current generators can be substituted by an ideal transformer with a suitable equivalent turn ratio.

Buck average model: CCM





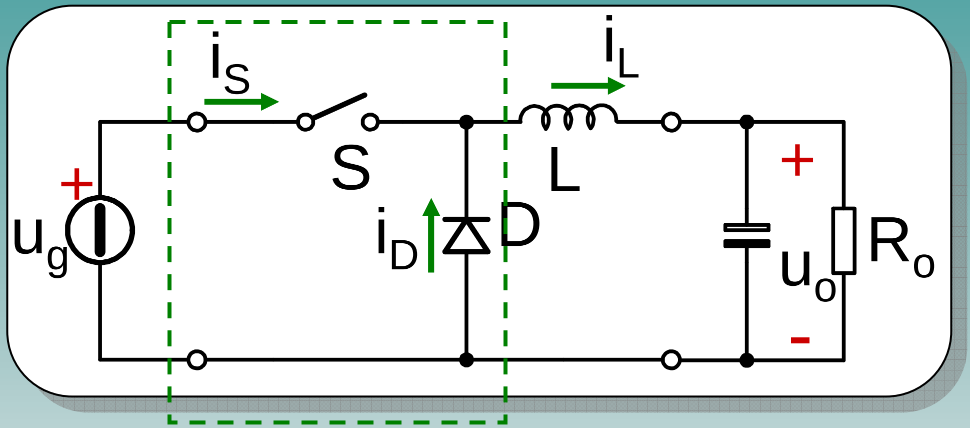
$$\bar{u}_D = \bar{u}_g - \bar{u}_S = \bar{u}_g - \frac{d'}{d}\bar{u}_D$$

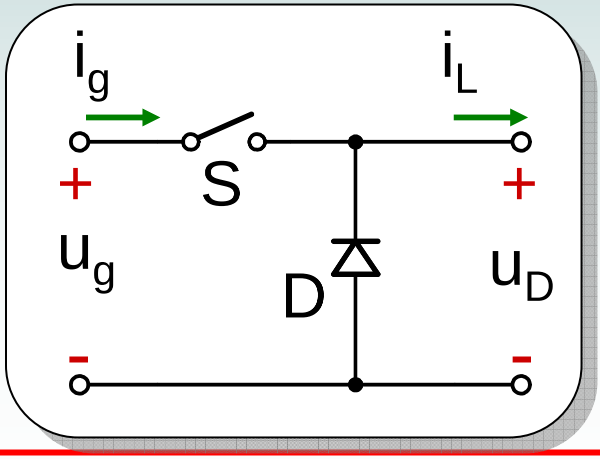
$$\bar{u}_D = d\,\bar{u}_g$$

$$\bar{i}_S = \frac{d}{d'}\bar{i}_D = \frac{d}{d'}\left(\bar{i}_L - \bar{i}_S\right)$$

$$\bar{i}_S = d\,\bar{i}_L$$

Buck average model (alternative approach): CCM



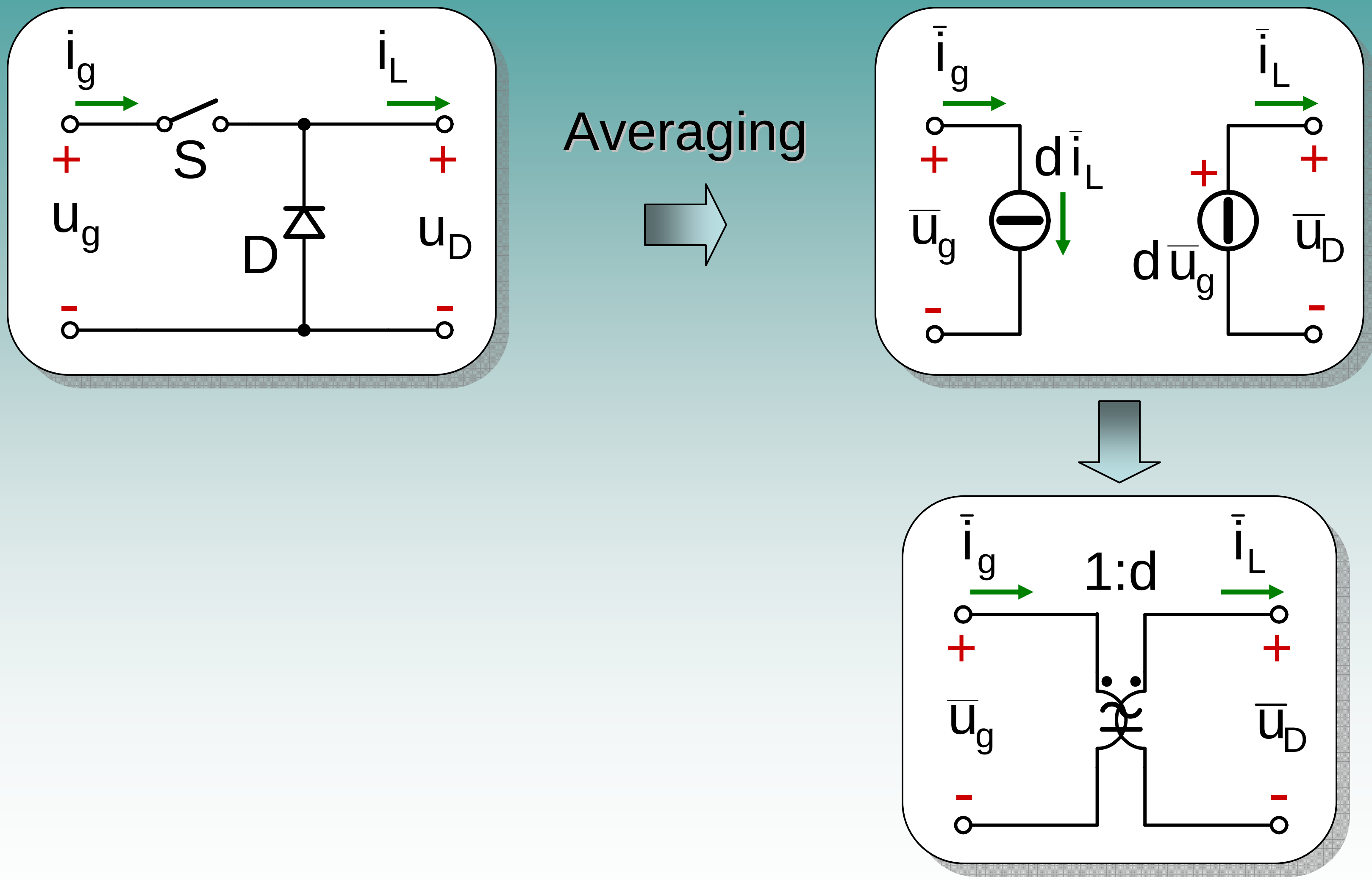


Switching cell

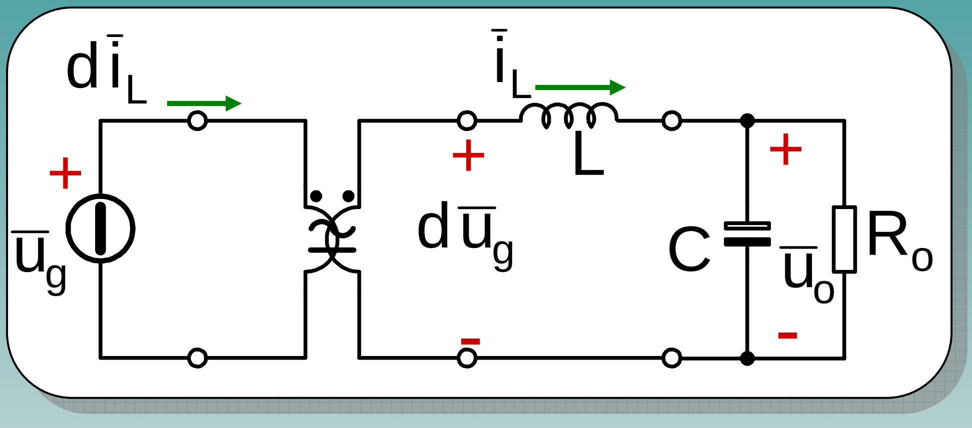
Independent variables: u_g, i_L

Dependent variables: u_D, i_g

Buck average model (alternative approach): CCM

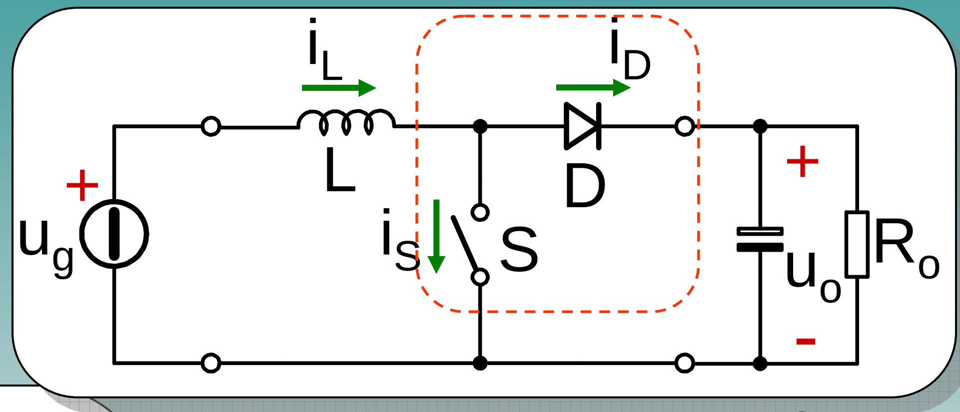


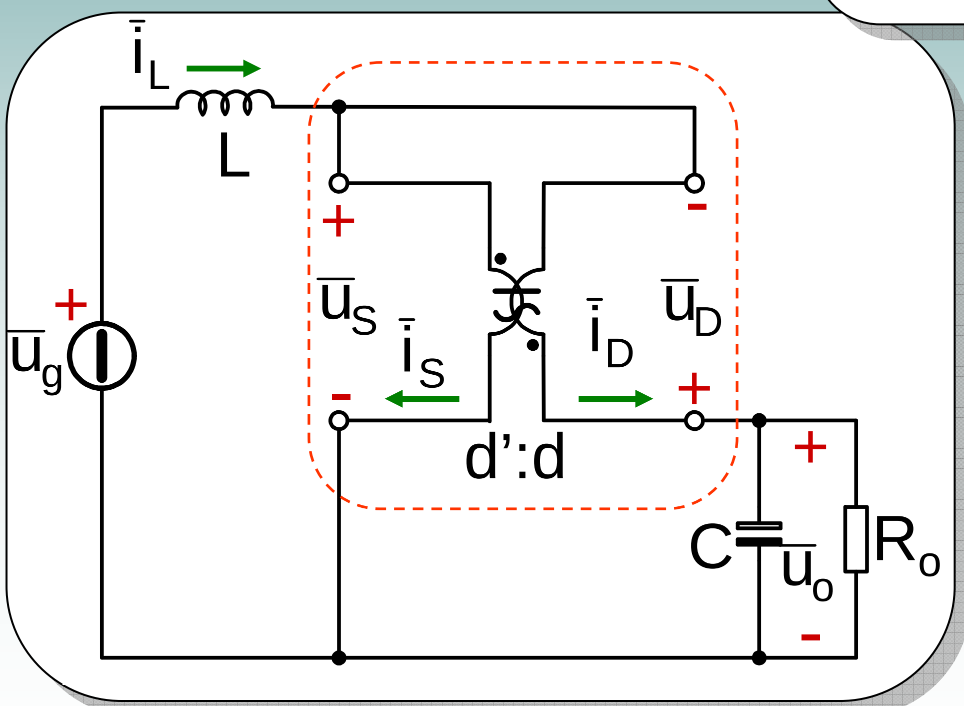
Buck average model: CCM



$$\begin{cases} L\dfrac{d\bar{i}_L}{dt} = \bar{u}_L = d\bar{u}_g - \bar{u}_o \\ C\dfrac{d\bar{u}_C}{dt} = \bar{i}_C = \bar{i}_L - \dfrac{\bar{u}_o}{R_o} \end{cases}$$

Boost average model: CCM





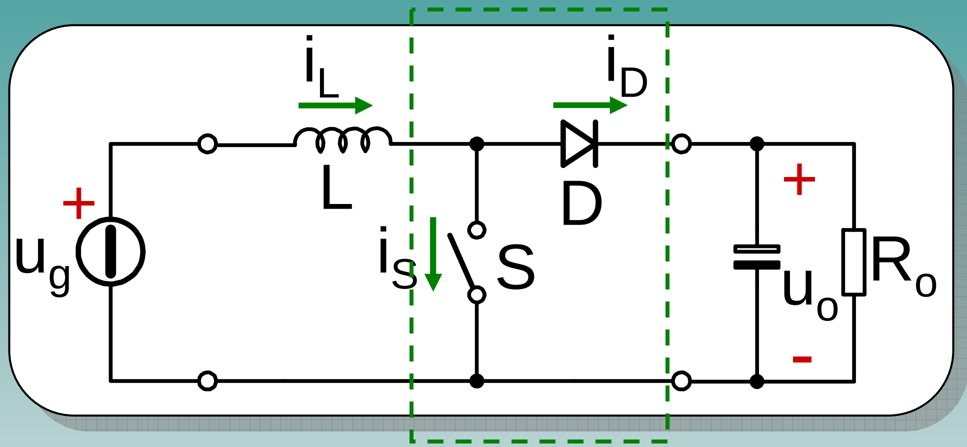
$$\bar{u}_S = \bar{u}_o - \bar{u}_D = \bar{u}_o - \frac{d}{d'}\bar{u}_S$$

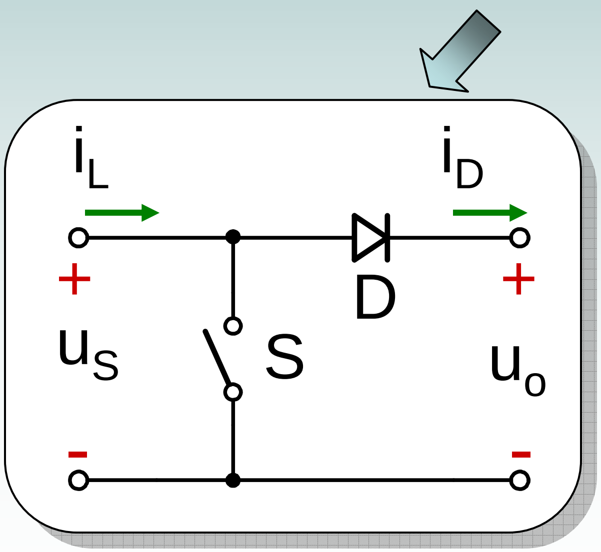
$$\bar{u}_S = d'\bar{u}_o$$

$$\bar{i}_D = \frac{d'}{d}\bar{i}_S = \frac{d'}{d}\left(\bar{i}_L - \bar{i}_D\right)$$

$$\bar{i}_D = d'\bar{i}_L$$

Boost average model (alternative approach): CCM



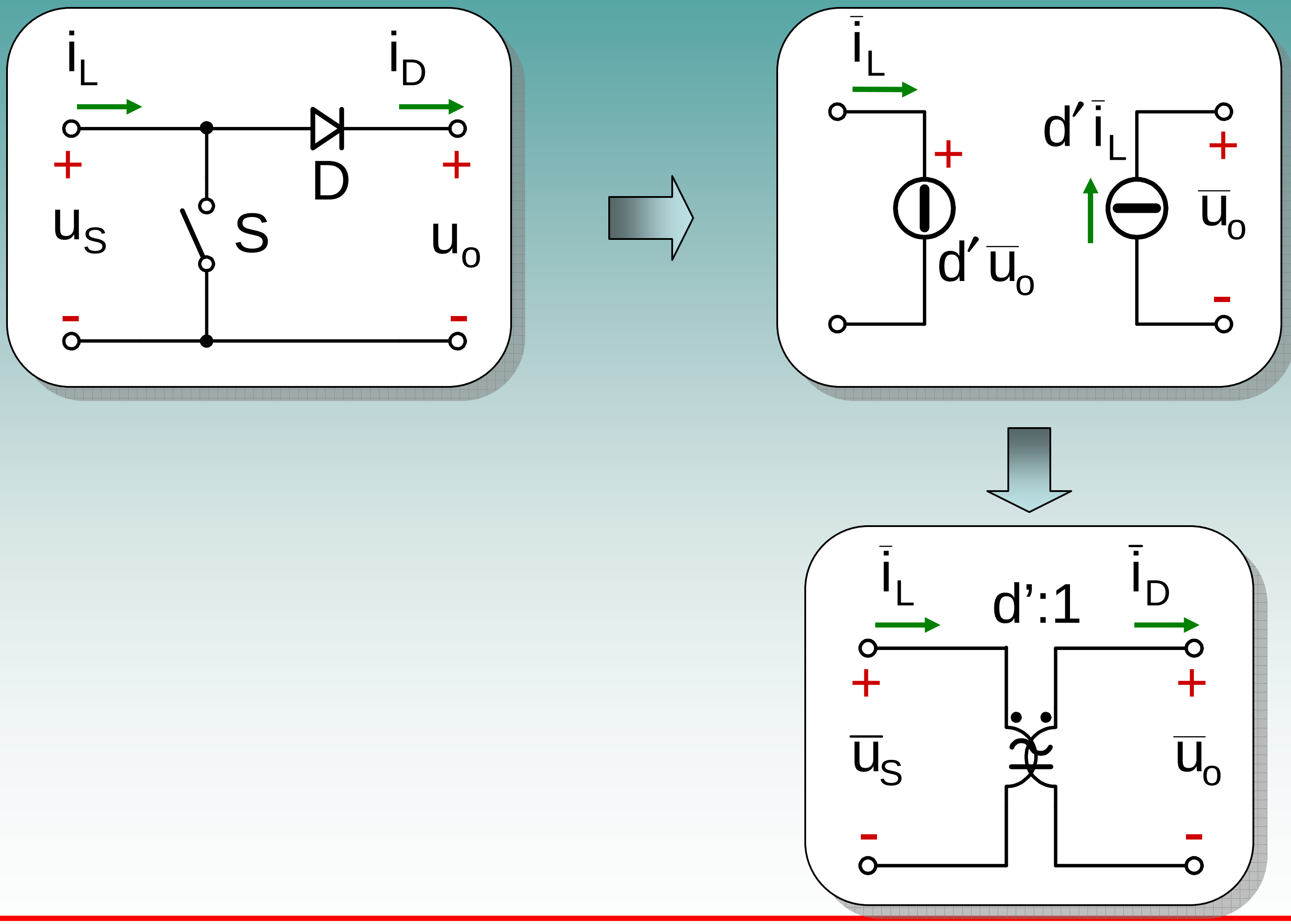


Switching cell

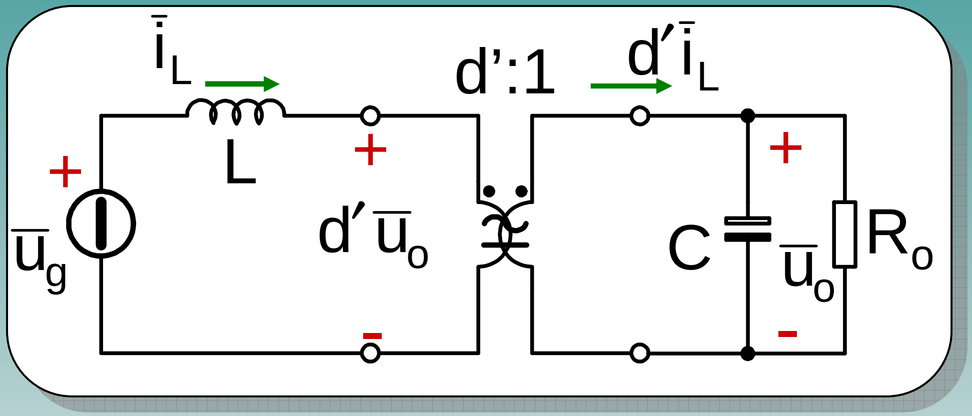
Independent variables: u_o, i_L

Dependent variables: u_S, i_D

Boost average model (alternative approach): CCM

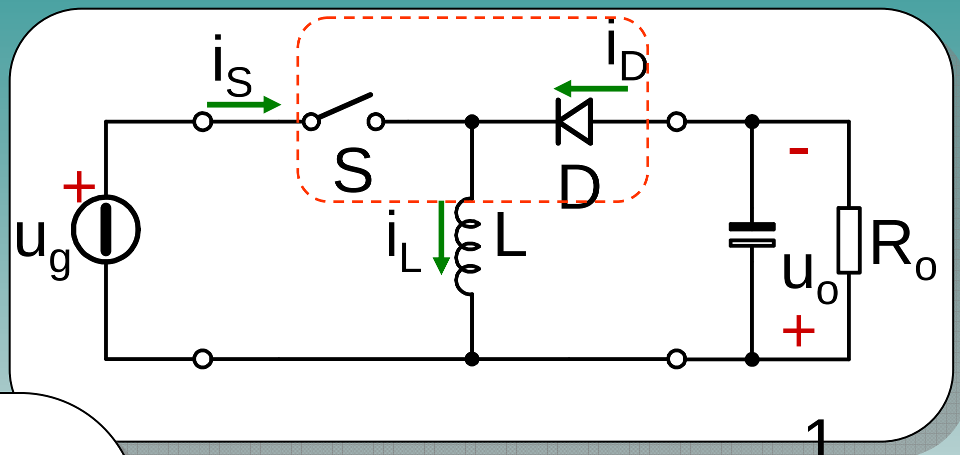


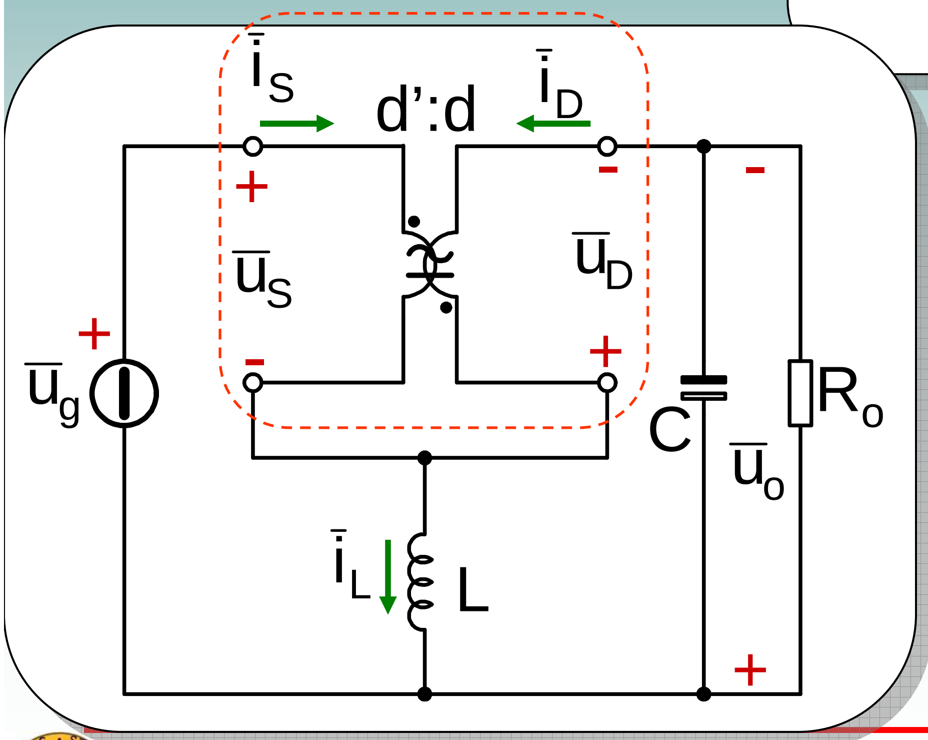
Boost average model: CCM



$$\begin{cases} L\dfrac{d\bar{i}_L}{dt} = \bar{u}_L = \bar{u}_g - d'\bar{u}_o \\ C\dfrac{d\bar{u}_C}{dt} = \bar{i}_C = d'\bar{i}_L - \dfrac{\bar{u}_o}{R_o} \end{cases}$$

Buck-Boost average model: CCM





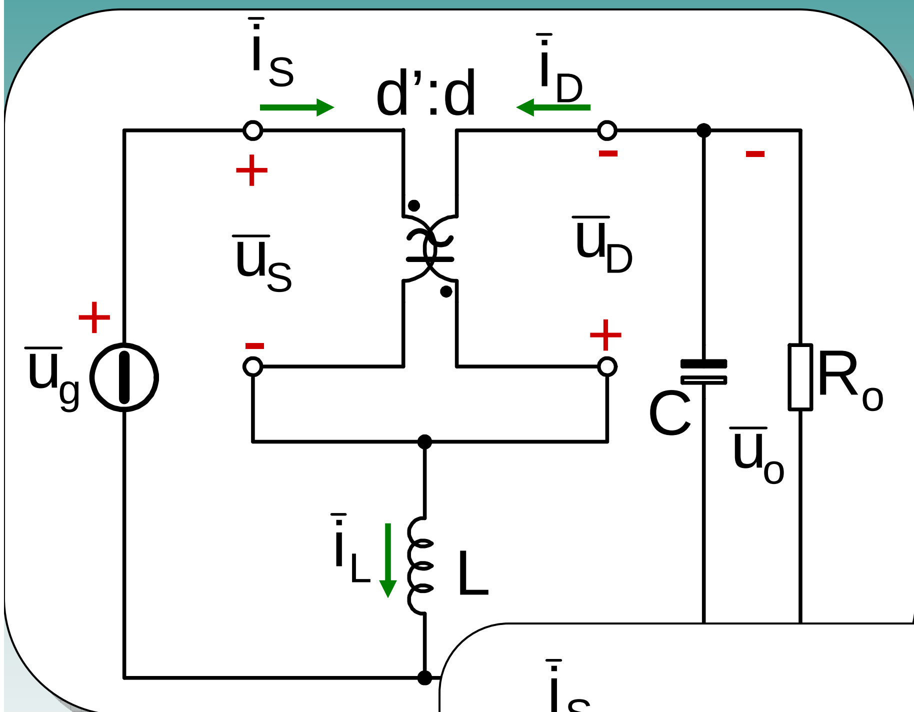
$$\bar{u}_g = \bar{u}_S + \bar{u}_D - \bar{u}_o = \frac{1}{d}\bar{u}_D - \bar{u}_o$$

$$\bar{u}_D = d(\bar{u}_g + \bar{u}_o) \quad \bar{u}_S = d'(\bar{u}_g + \bar{u}_o)$$

$$\bar{i}_D = \frac{d'}{d}\bar{i}_S = \frac{d'}{d}(\bar{i}_L - \bar{i}_D)$$

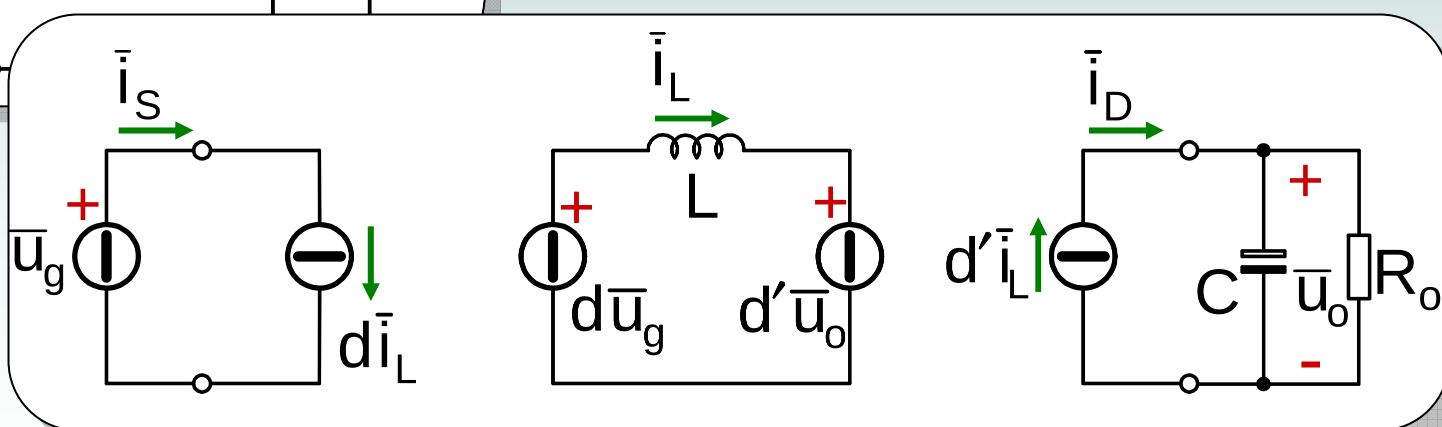
$$\bar{i}_D = d'\bar{i}_L \qquad \bar{i}_S = d\bar{i}_L$$

Buck-Boost average model: CCM

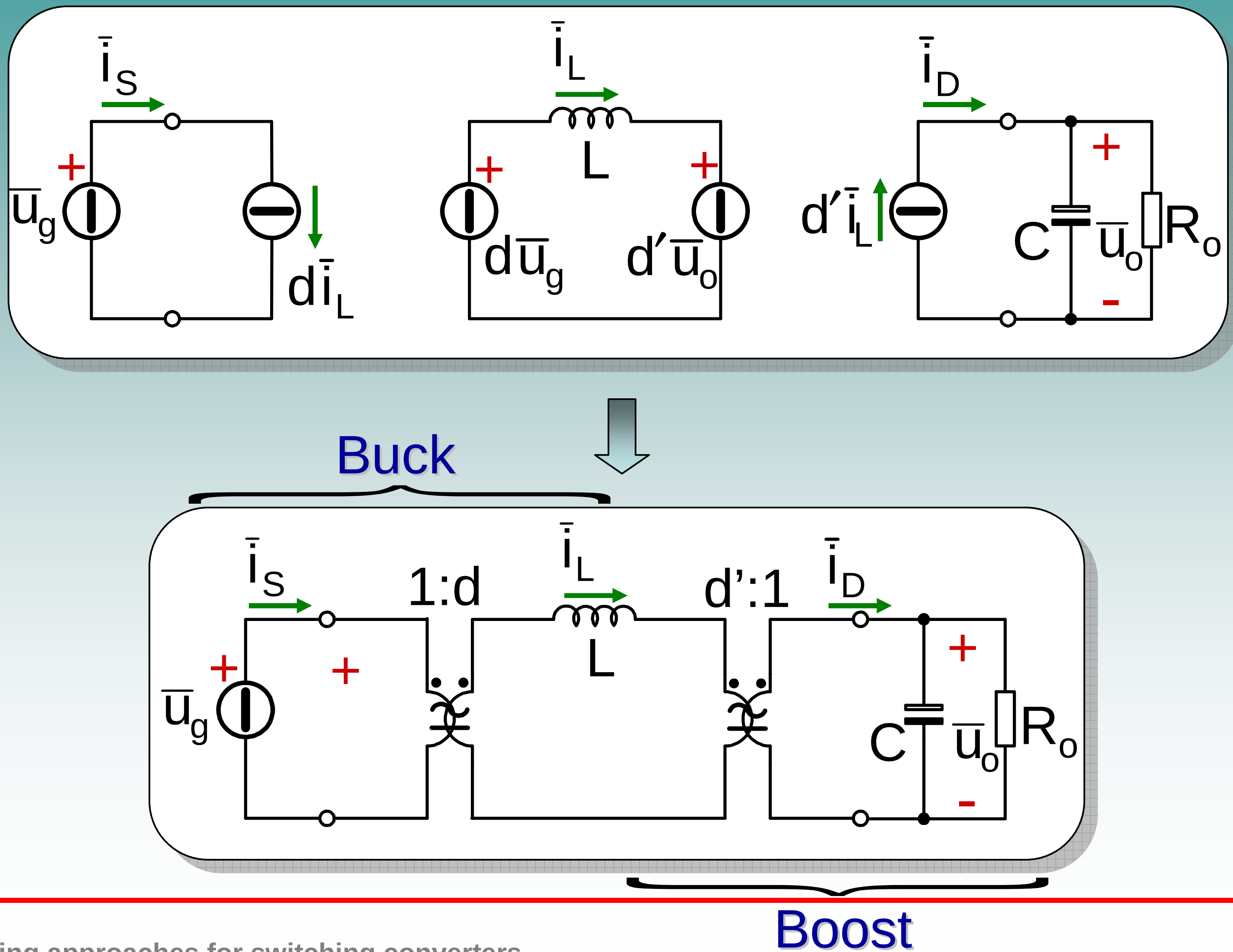


$$\bar{u}_D = d\left(\bar{u}_g + \bar{u}_o\right)$$

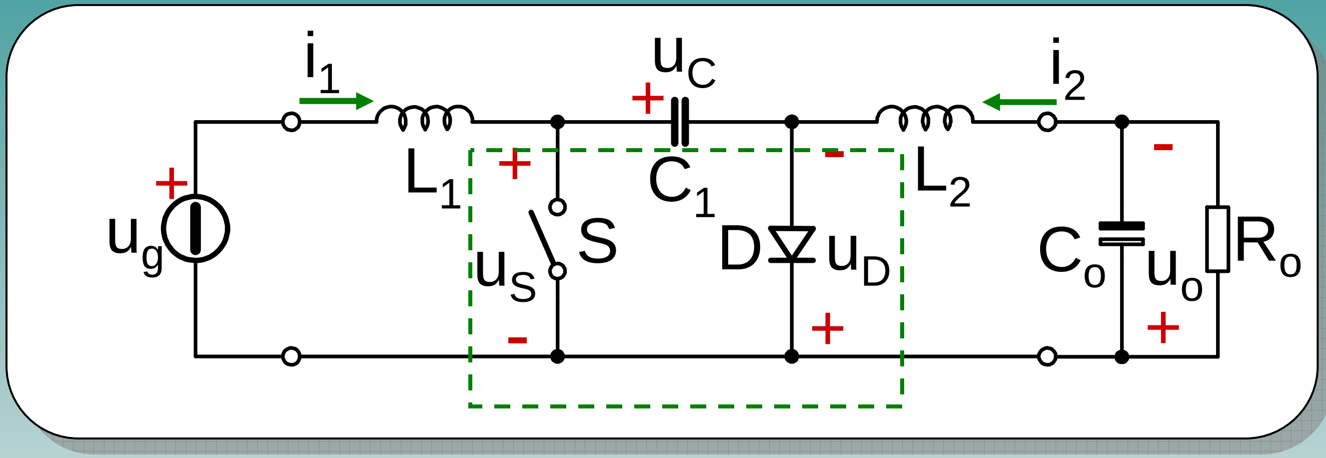
$$\begin{cases} L\dfrac{d\bar{i}_L}{dt} = \bar{u}_L = \bar{u}_D - \bar{u}_o = d\bar{u}_g - d'\bar{u}_o \\ C\dfrac{d\bar{u}_C}{dt} = \bar{i}_C = d'\bar{i}_L - \dfrac{\bar{u}_o}{R_o} \end{cases}$$



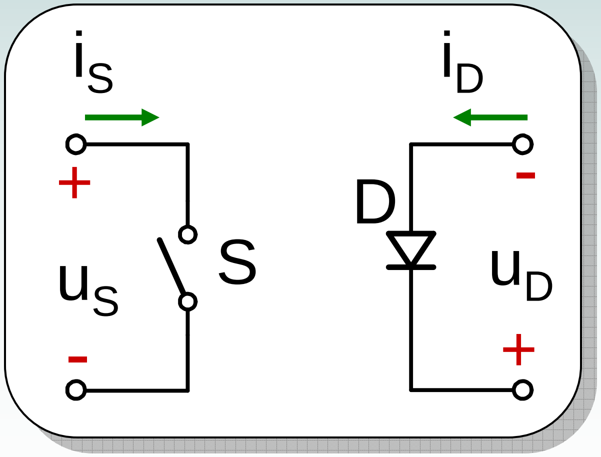
Buck-Boost equivalent average model: CCM



Cuk average model: CCM



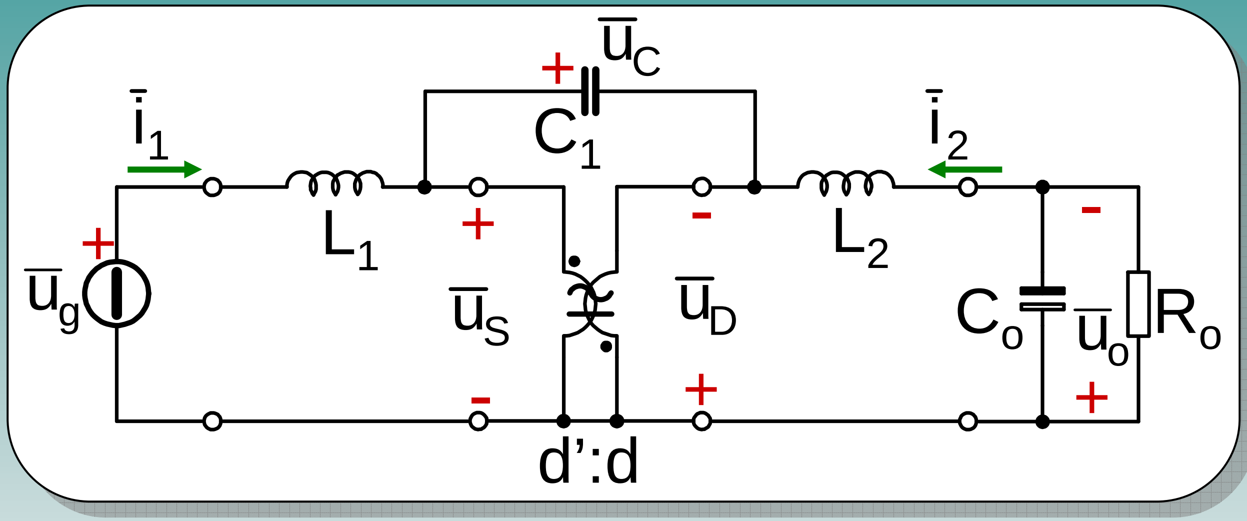
Switching cell



$$\begin{cases} \bar{u}_S = d'\bar{u}_C \\ \bar{u}_D = d\bar{u}_C \end{cases} \Rightarrow \bar{u}_S = \frac{d'}{d}\bar{u}_D$$

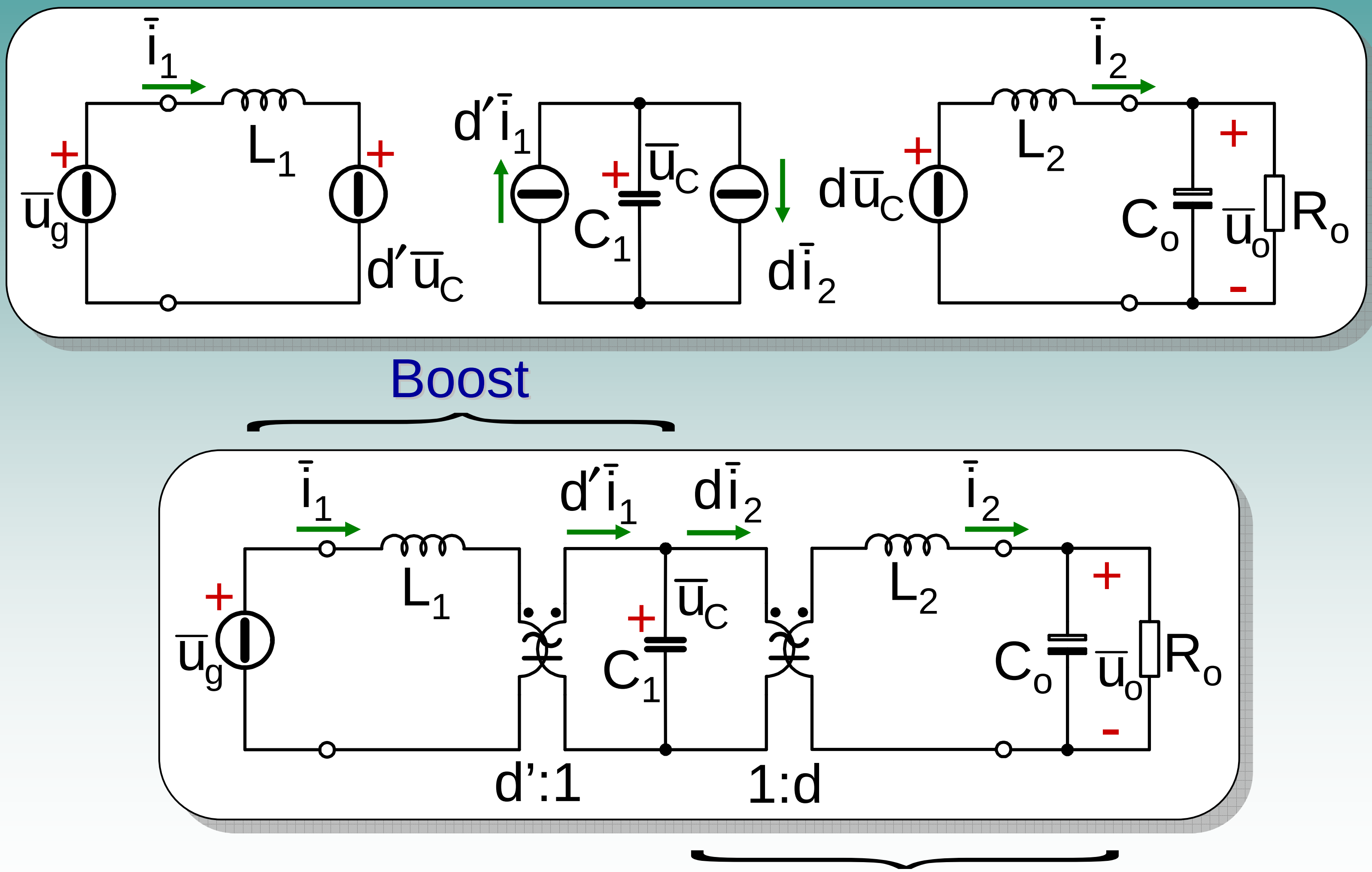
$$\begin{cases} \bar{i}_S = d\left(\bar{i}_1 + \bar{i}_2\right) \\ \bar{i}_D = d'\left(\bar{i}_1 + \bar{i}_2\right) \end{cases} \Rightarrow \bar{i}_S = \frac{d}{d'}\bar{i}_D$$

Cuk average model: CCM



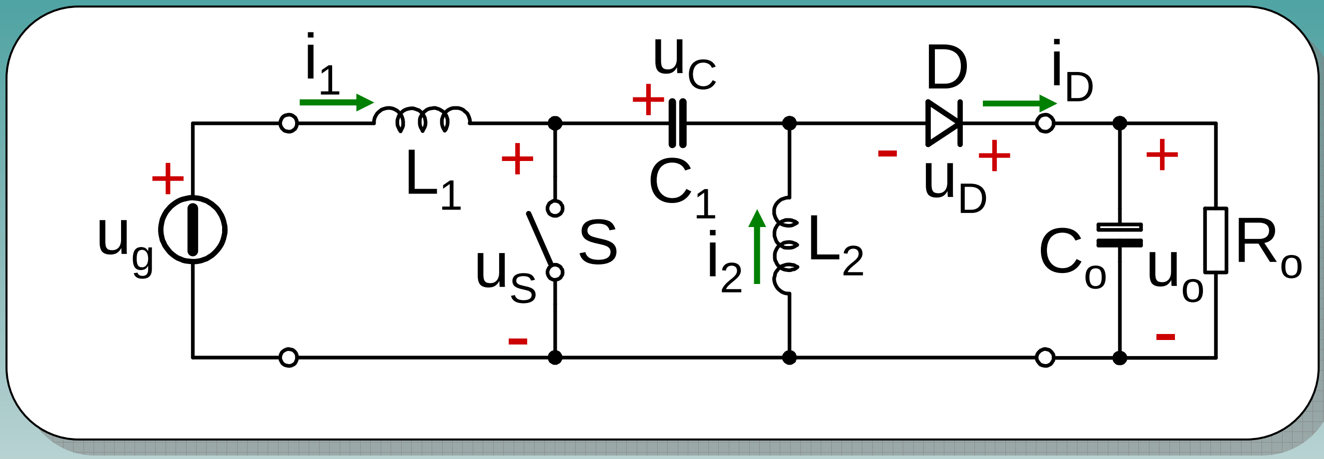
$$
\begin{cases}
L_1 \dfrac{d\bar{i}_1}{dt} = \bar{u}_g - d'\bar{u}_C \\
L_2 \dfrac{d\bar{i}_2}{dt} = d\bar{u}_C - \bar{u}_o \\
C_1 \dfrac{d\bar{u}_C}{dt} = d'\bar{i}_1 - d\bar{i}_2 \\
C_o \dfrac{d\bar{u}_o}{dt} = \bar{i}_2 - \dfrac{\bar{u}_o}{R_o}
\end{cases}
$$

Cuk average model: CCM

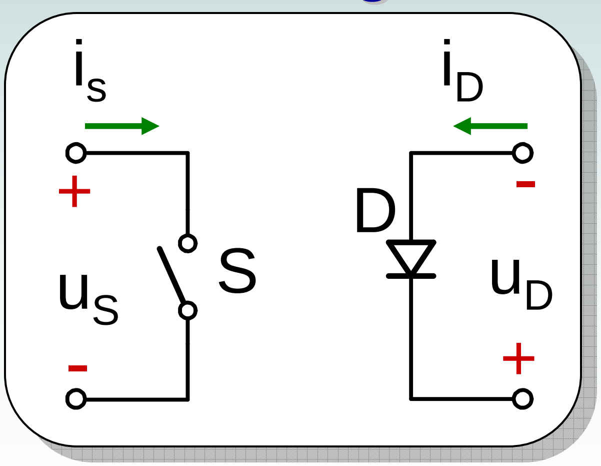


Buck

SEPIC average model: CCM



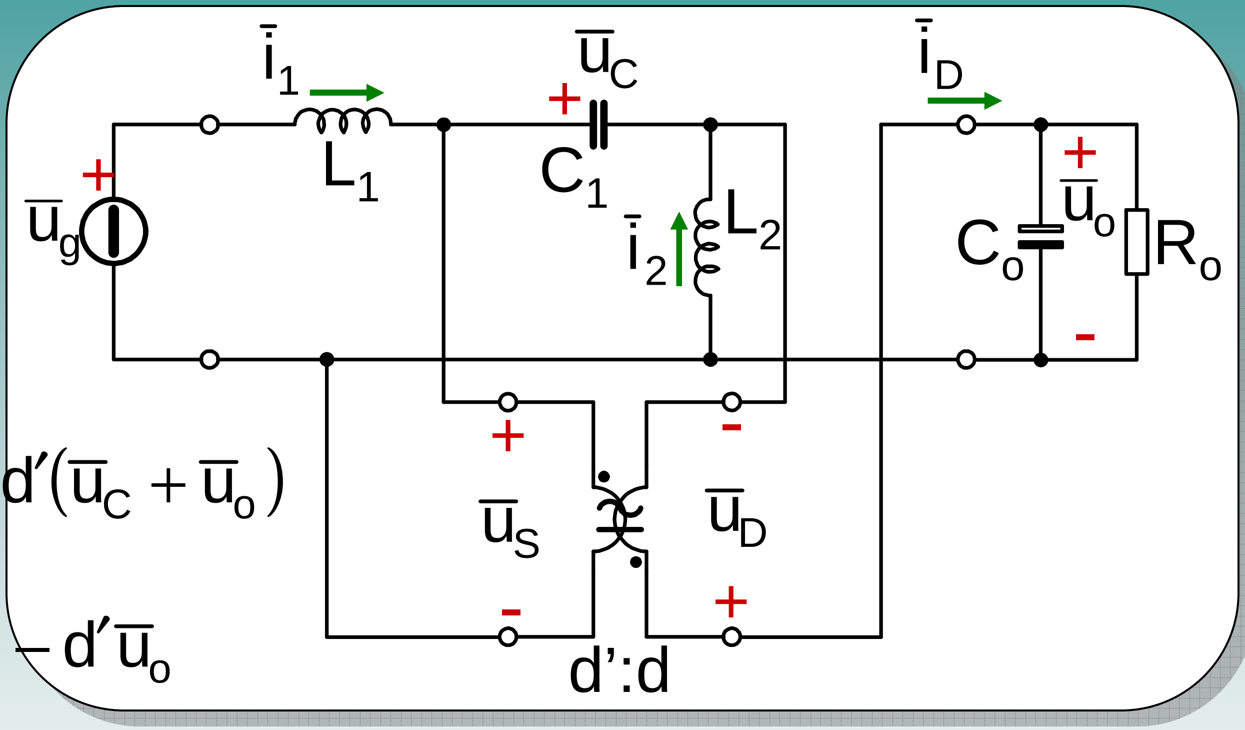
Switching cell



$$\begin{cases}\overline{u}_S = d'\left(\overline{u}_C + \overline{u}_o\right)\\ \overline{u}_D = d\left(\overline{u}_C + \overline{u}_o\right)\end{cases} \Rightarrow \quad \overline{u}_S = \frac{d'}{d}\overline{u}_D$$

$$\begin{cases}\bar{i}_S = d\left(\bar{i}_1 + \bar{i}_2\right)\\ \bar{i}_D = d'\left(\bar{i}_1 + \bar{i}_2\right)\end{cases} \Rightarrow \quad \bar{i}_S = \frac{d}{d'}\bar{i}_D$$

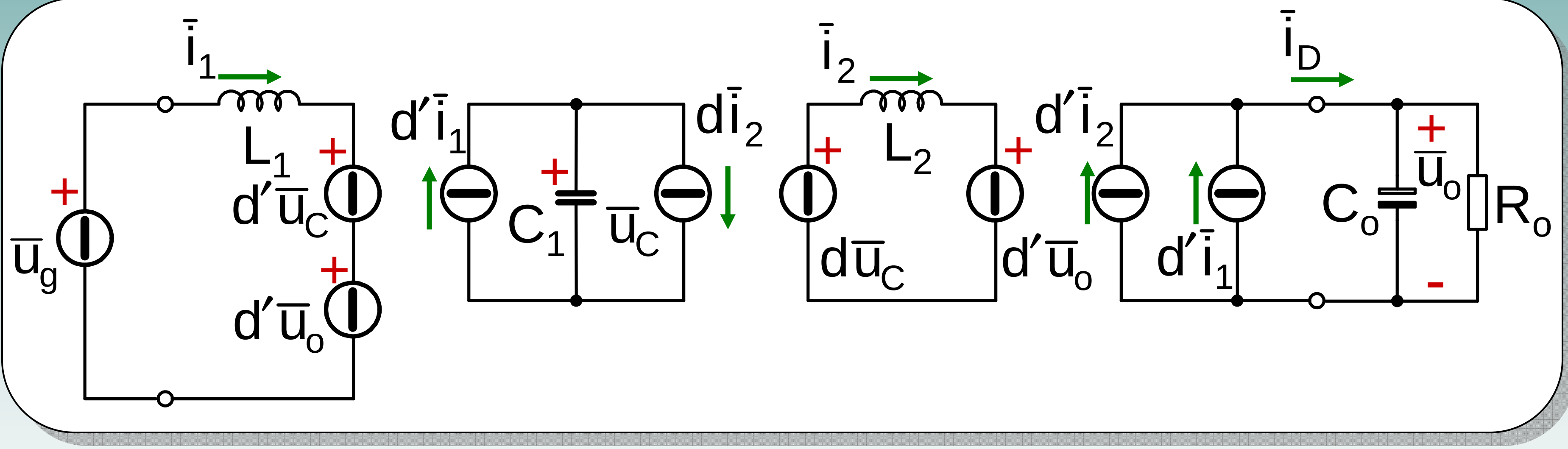
SEPIC average model: CCM



$$\begin{cases} L_1 \dfrac{d\bar{i}_1}{dt} = \bar{u}_g - d'(\bar{u}_C + \bar{u}_o) \\ L_2 \dfrac{d\bar{i}_2}{dt} = d\bar{u}_C - d'\bar{u}_o \\ C_1 \dfrac{d\bar{u}_C}{dt} = d'\bar{i}_1 - d\bar{i}_2 \\ C_o \dfrac{d\bar{u}_o}{dt} = d'(\bar{i}_1 + \bar{i}_2) - \dfrac{\bar{u}_o}{R_o} \end{cases}$$

SEPIC average model: CCM

Alternative approach



Model perturbation

Generic voltage or current: $\bar{x} = X + \hat{x}$

Small-signal approximation: $\hat{x} << X$

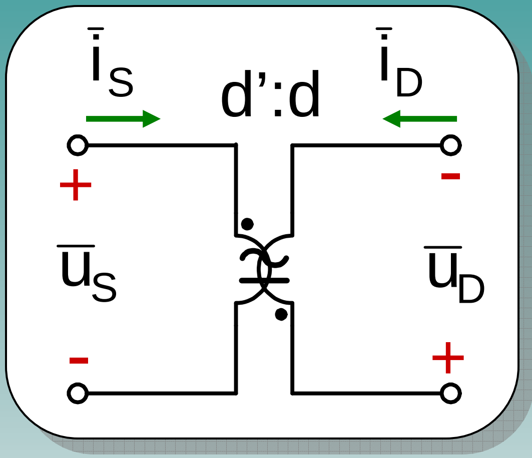
Product of variables: $\bar{x} \cdot \bar{y} \approx XY + X\hat{y} + \hat{x}Y$

Examples:

$$d\bar{i}_L = \left(D + \hat{d}\right)\left(I_L + \hat{i}_L\right) \approx DI_L + D\hat{i}_L + \hat{d}I_L$$

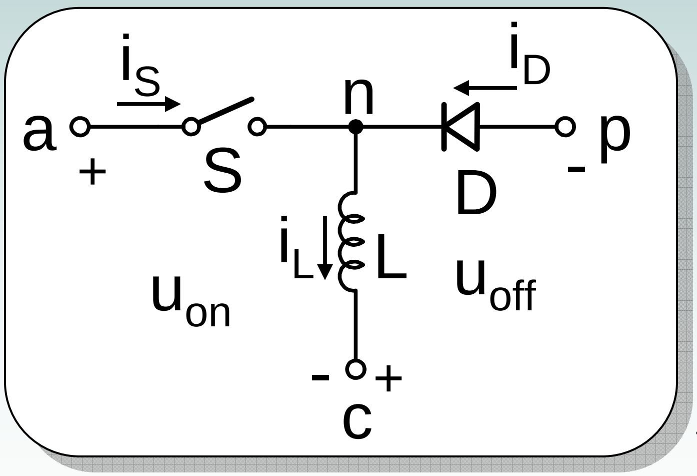
$$d\bar{u}_g = \left(D + \hat{d}\right)\left(U_g + \hat{u}_g\right) \approx DU_g + D\hat{u}_g + \hat{d}U_g$$

General switching cell: DC and small-signal model



$$D(U_S + \hat{u}_S) + \hat{d}U_S \approx D'(U_D + \hat{u}_D) - \hat{d}U_D$$

$$\underbrace{U_S + \hat{u}_S}_{\bar{u}_S} + \hat{d}\left(\frac{U_S + U_D}{D}\right) \approx \frac{D'}{D}(\underbrace{U_D + \hat{u}_D}_{\bar{u}_D})$$

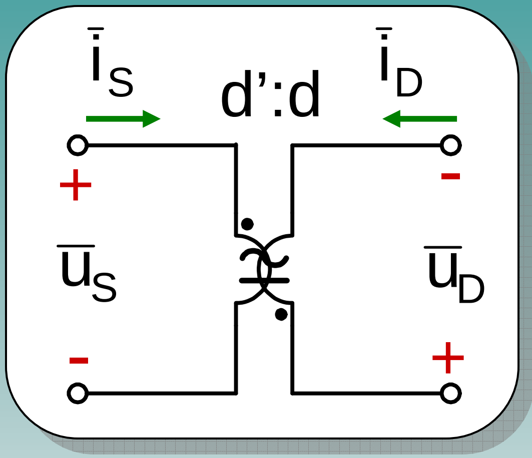


At steady-state:

$$\bar{u}_L = 0 \quad \Rightarrow \quad \begin{cases} \bar{u}_S = U_S = U_{on} \\ \bar{u}_D = U_D = U_{off} \end{cases}$$

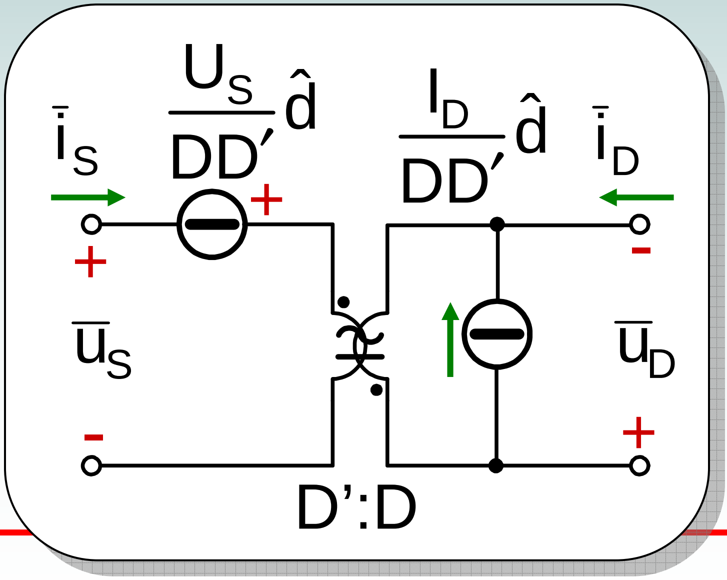
$$\frac{U_S + U_D}{D} = \frac{U_S}{D}\left(1 + \frac{U_D}{U_S}\right) = \frac{U_S}{D}\left(1 + \frac{D}{D'}\right) = \frac{U_S}{DD'}$$

General switching cell: DC and small-signal model

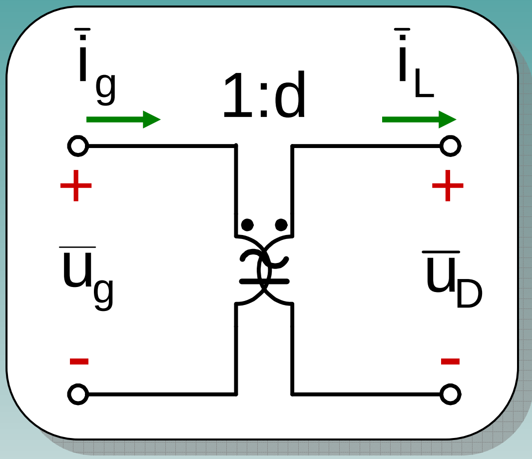


$$D'\left(I_S + \hat{i}_S\right) - \hat{d}I_S \approx D\left(I_D + \hat{i}_D\right) + \hat{d}I_D$$

$$\underbrace{I_D + \hat{i}_D}_{\bar{i}_D} \approx \frac{D'}{D}\underbrace{\left(I_S + \hat{i}_S\right)}_{\bar{i}_S} - \hat{d}\left(\frac{I_S + I_D}{D}\right)$$

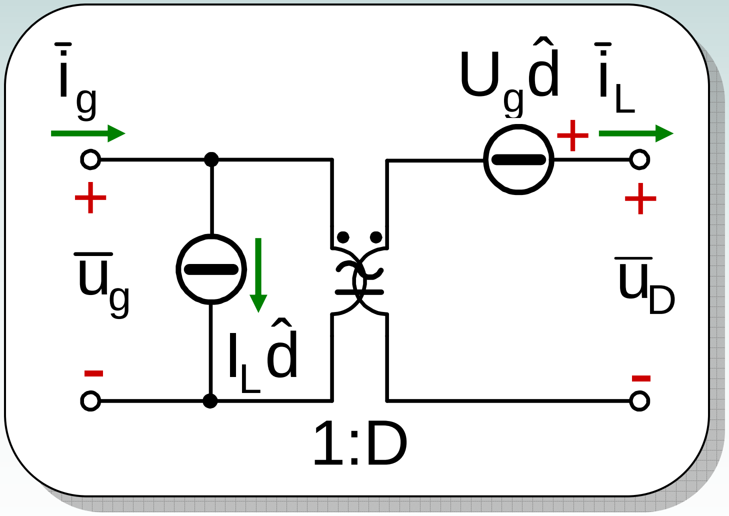


Buck switching cell: DC and small-signal model

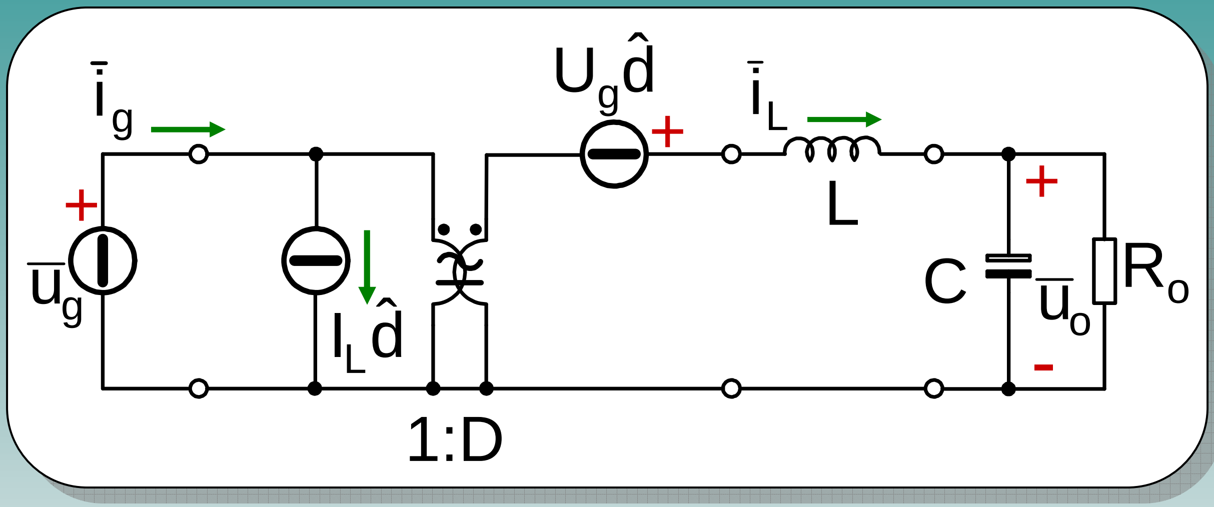


Perturbation and linearization:

$$\begin{cases} \bar{i}_g = D\bar{i}_L + I_L\hat{d} \\ \bar{u}_D = D\bar{u}_g + U_g\hat{d} \end{cases}$$



Buck DC and small-signal model

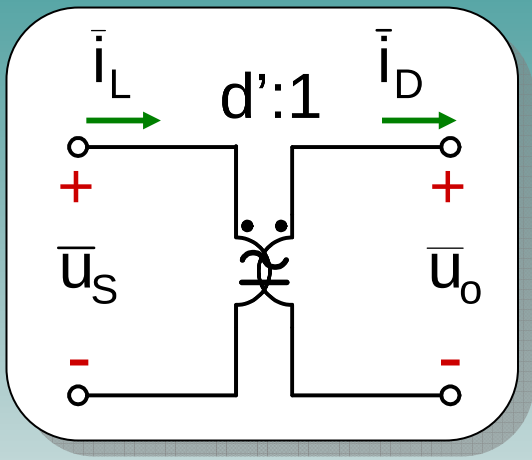


Duty-cycle to output voltage transfer function:

$$G_{ud}(s) = \frac{\hat{U}_o(s)}{\hat{D}(s)} = \frac{U_g}{1 + \frac{s}{Q\omega_o} + \frac{s^2}{\omega_o^2}}$$

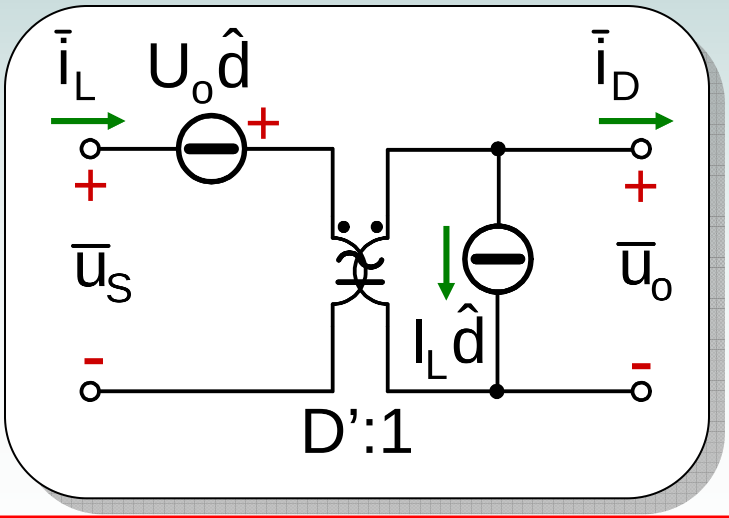
$$\omega_o = \frac{1}{\sqrt{LC}} \qquad Q = R_o\sqrt{\frac{C}{L}}$$

Boost switching cell: DC and small-signal model

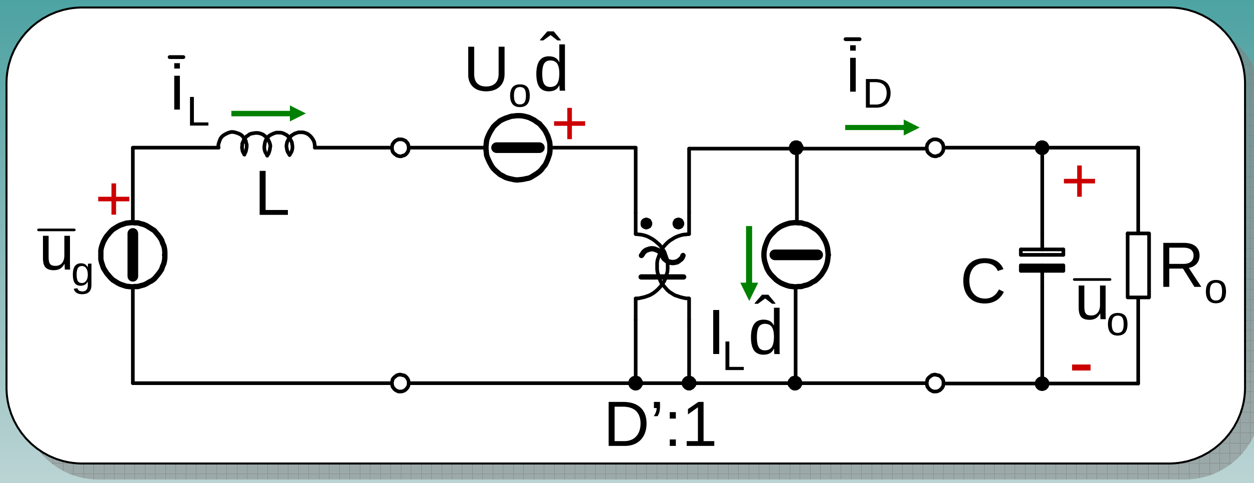


Perturbation and linearization:

$$\begin{cases} \bar{i}_D = D'\bar{i}_L - I_L\hat{d} \\ \bar{u}_S = D'\bar{u}_o - U_o\hat{d} \end{cases}$$



Boost DC and small-signal model



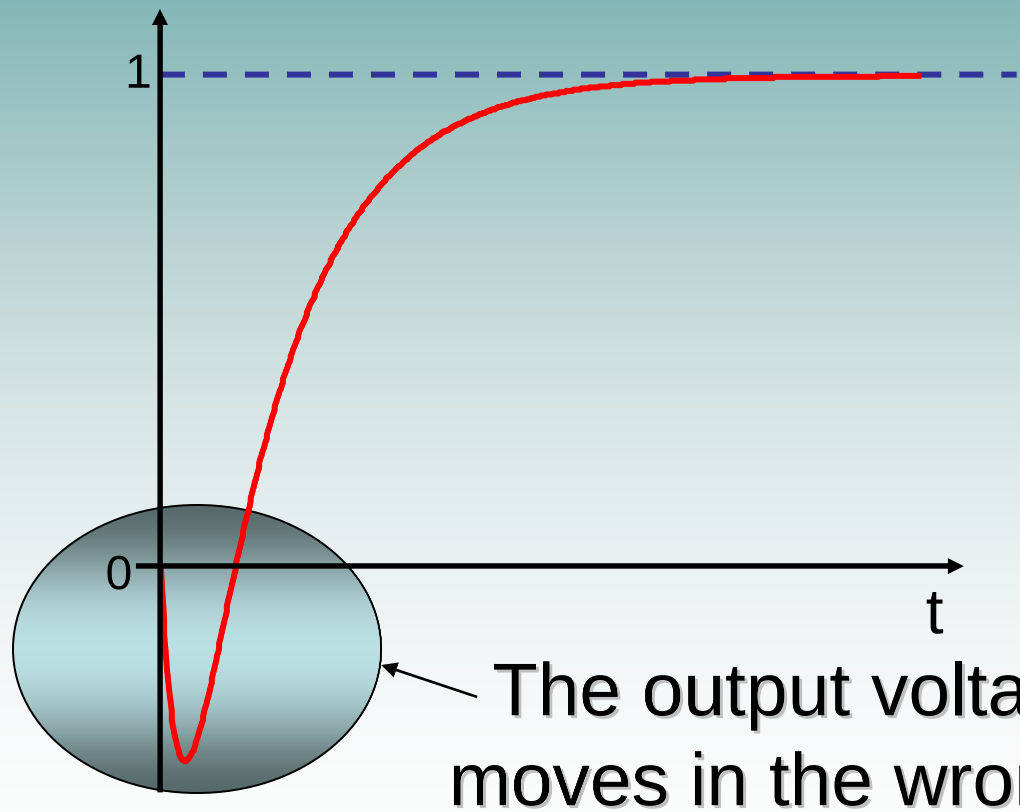
Duty-cycle to output voltage transfer function:

$$G_{ud}(s) = U_g M^2 \frac{1 - s\frac{L}{R_o}M^2}{1 + s\frac{L}{R_o}M^2 + s^2 LCM^2}$$

RHP zero

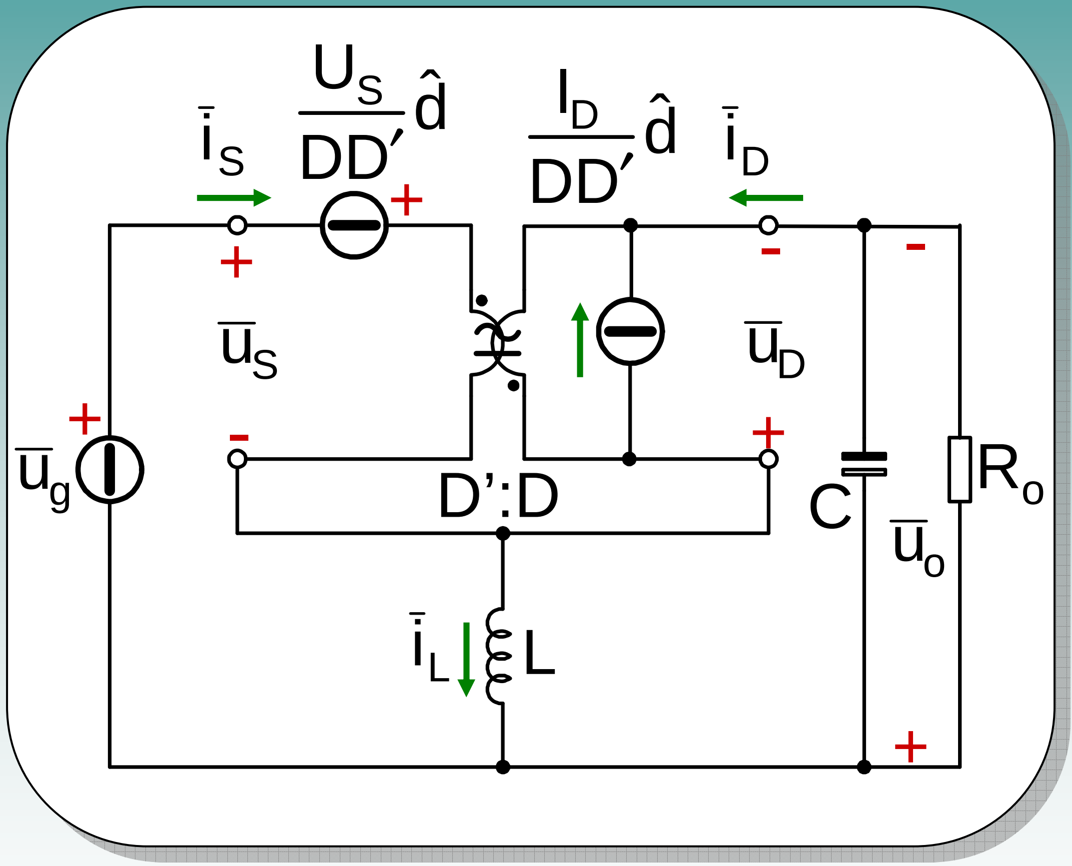
Boost small-signal model: CCM

Normalized output voltage response to a duty-cycle step change:

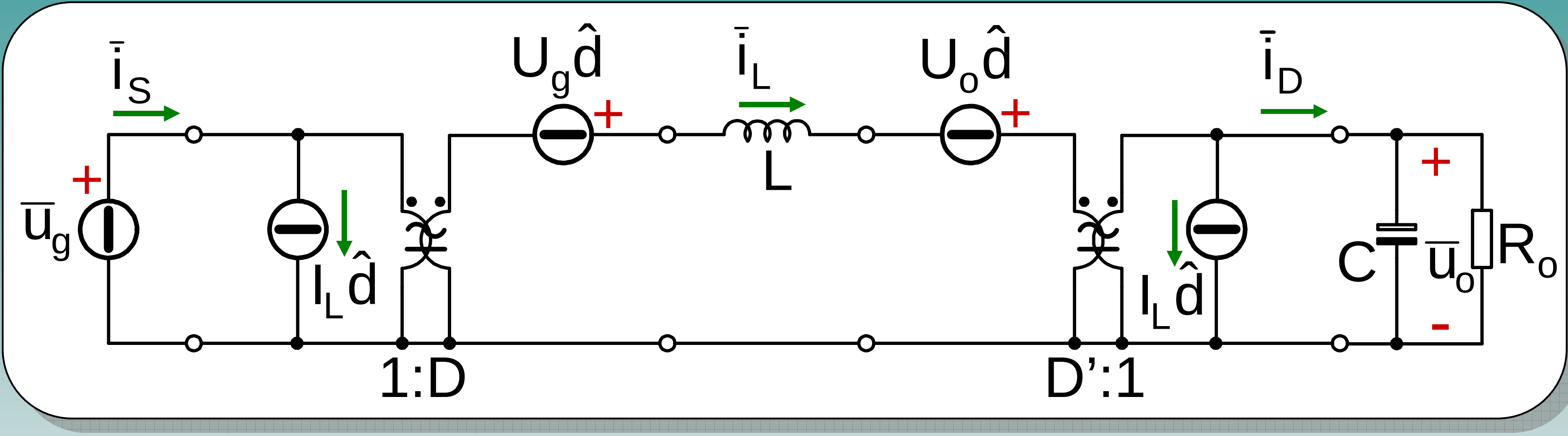


The output voltage initially moves in the wrong direction

Buck-Boost DC and small-signal model



Buck-Boost DC and small-signal model

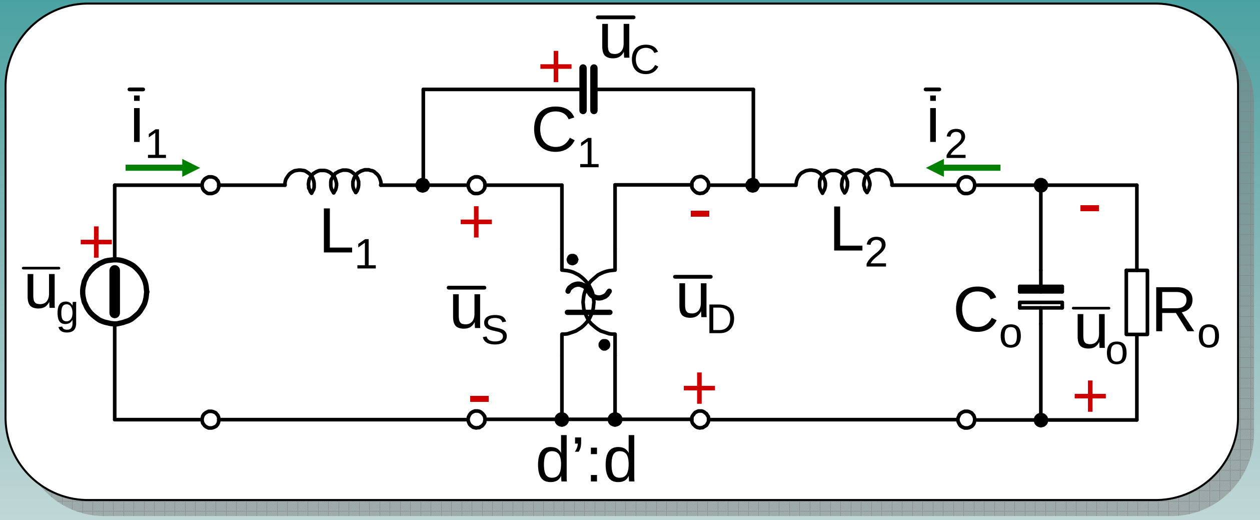


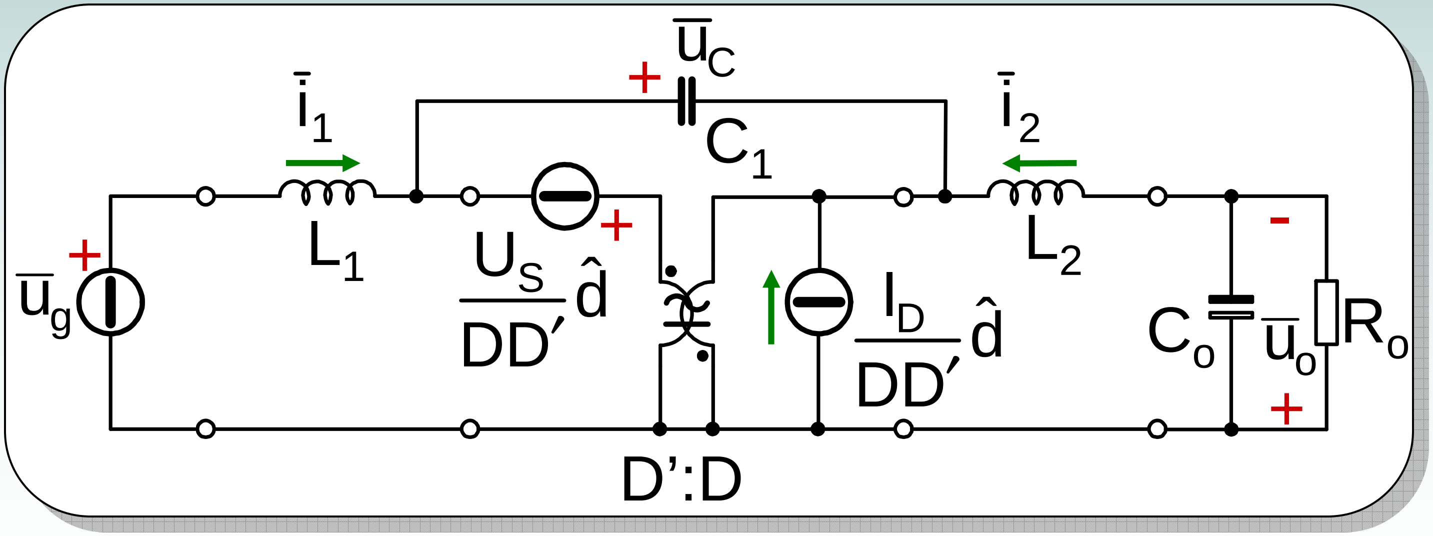
Duty-cycle to output voltage transfer function:

$$G_{ud}(s) = U_g(1+M)^2 \frac{1 - s\frac{L}{R_o}M(1+M)}{1 + s\frac{L}{R_o}(1+M)^2 + s^2LC(1+M)^2}$$

RHP zero

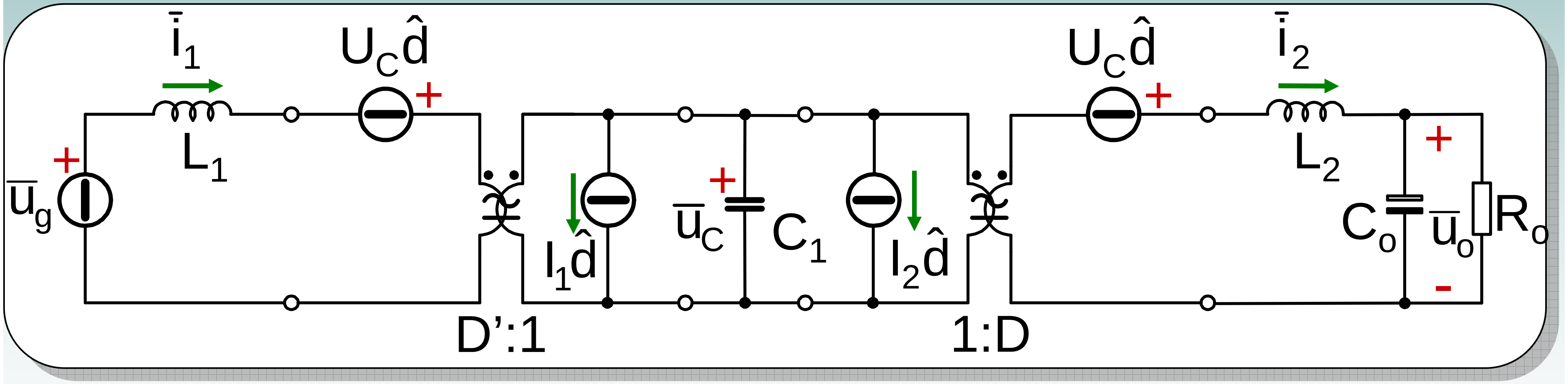
Cuk DC and small-signal model



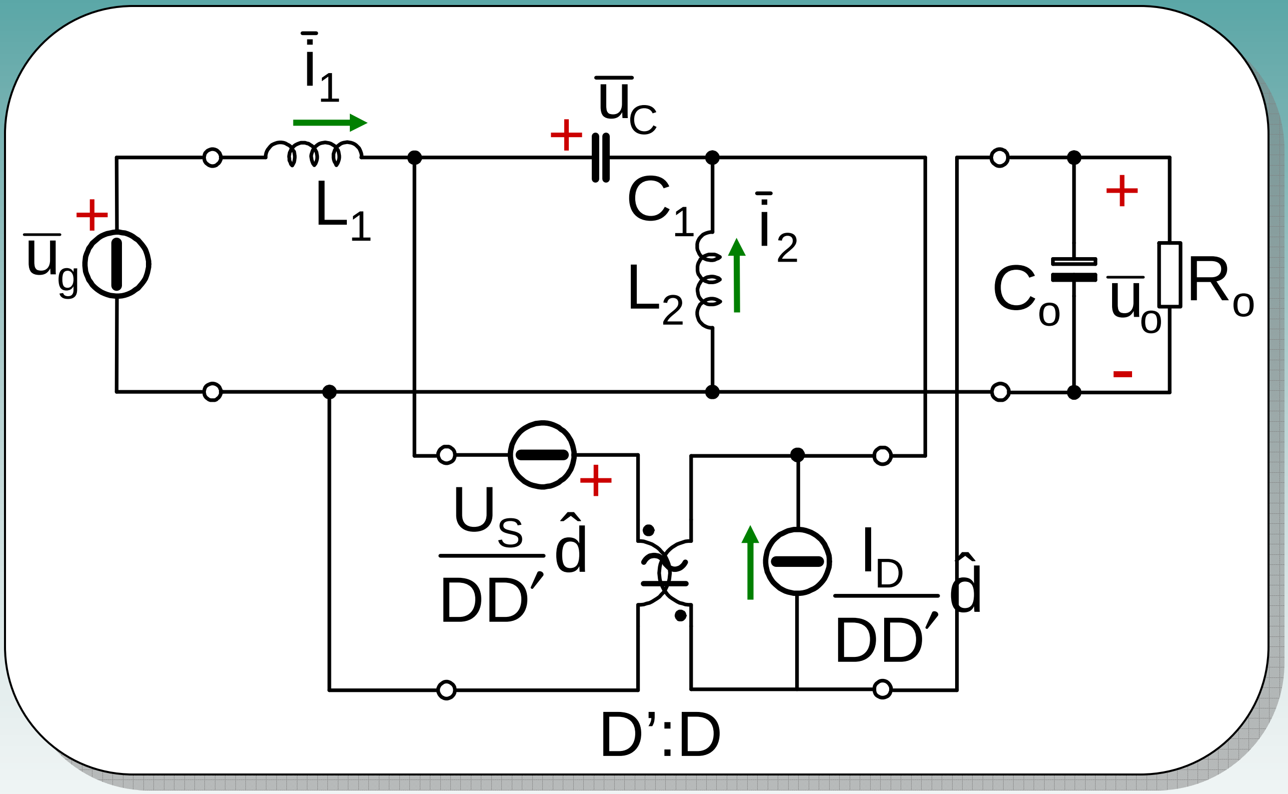


Cuk DC and small-signal model

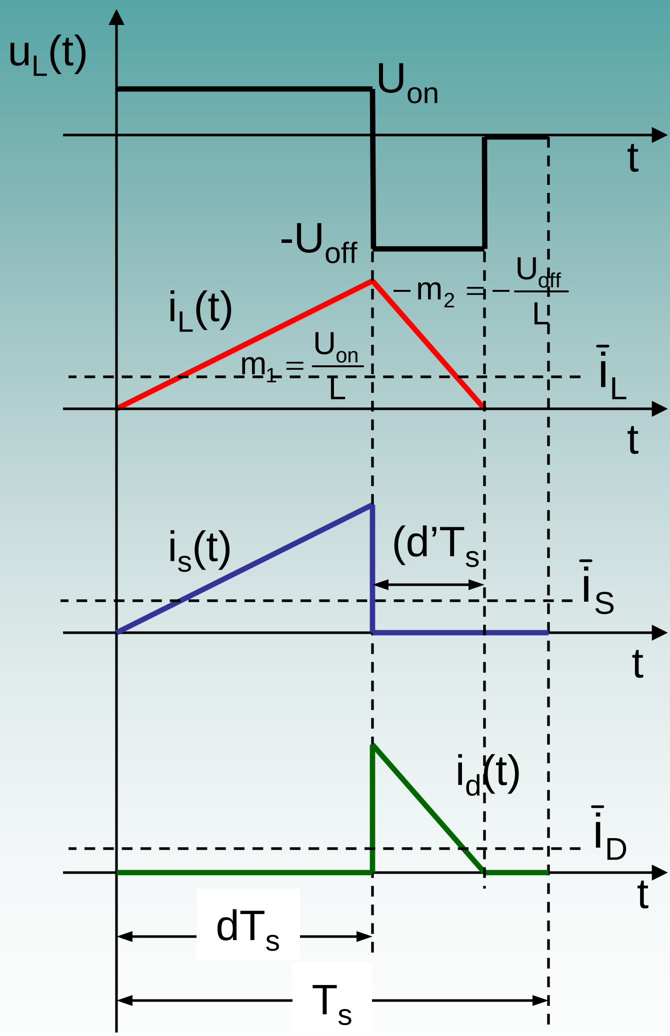
Alternative approach



SEPIC DC and small-signal model



Discontinuous conduction mode - DCM



At steady-state:

$$\overline{u}_L = 0 \quad \Rightarrow \quad \overline{u}_{on} d T_S = \overline{u}_{off} d' T_S$$

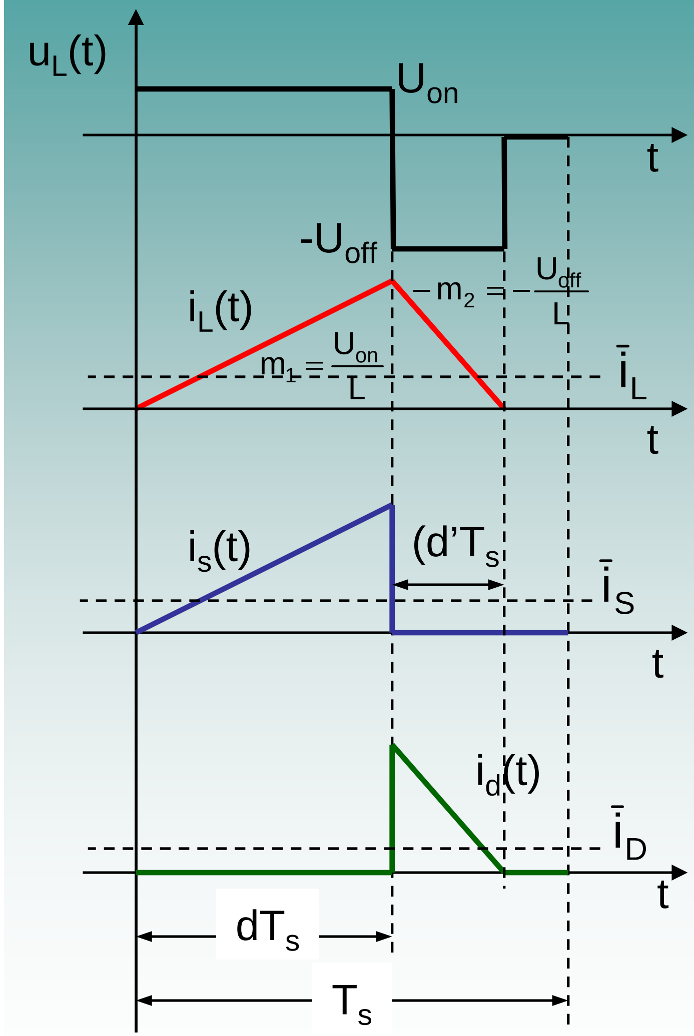
$$\frac{\overline{u}_{on}}{\overline{u}_{off}} = \frac{d'}{d}$$

$$\overline{i}_L = \frac{i_{Lpk}}{2}(d + d') = \frac{d^2}{2Lf_S}\overline{u}_{on}\left(1 + \frac{\overline{u}_{on}}{\overline{u}_{off}}\right)$$

$$\overline{i}_D = \frac{i_{Lpk}}{2} d' = \frac{d^2}{2Lf_S}\frac{\overline{u}_{on}^2}{\overline{u}_{off}}$$

$$\overline{i}_S = \frac{i_{Lpk}}{2} d = \frac{d^2}{2Lf_S}\overline{u}_{on}$$

Discontinuous conduction mode - DCM



$$I_{oN} = \frac{I_o}{I_N}, \quad I_N = \frac{U_g}{2Lf_s}$$

Buck:

$$I_o = \bar{i}_L = \frac{d^2}{2Lf_S} U_g \left(\frac{1}{M} - 1 \right) \Rightarrow M = \frac{1}{1 + \frac{I_{oN}}{d^2}}$$

Boost:

$$I_o = \bar{i}_D = \frac{d^2}{2Lf_S} U_g \left(\frac{1}{M-1} \right) \Rightarrow M = 1 + \frac{d^2}{I_{oN}}$$

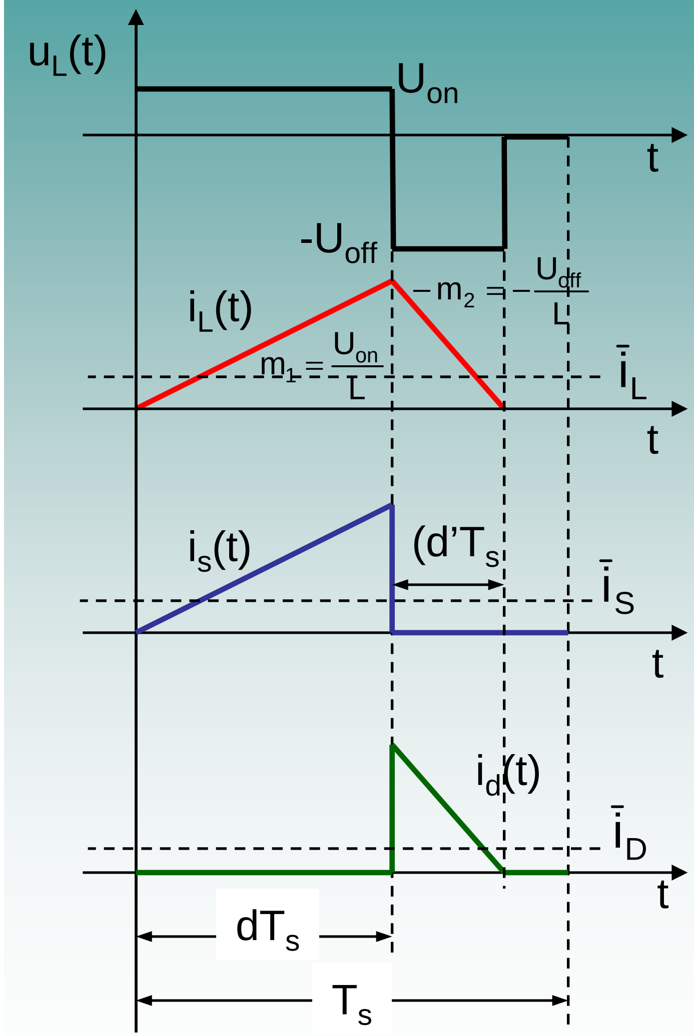
Buck-Boost:

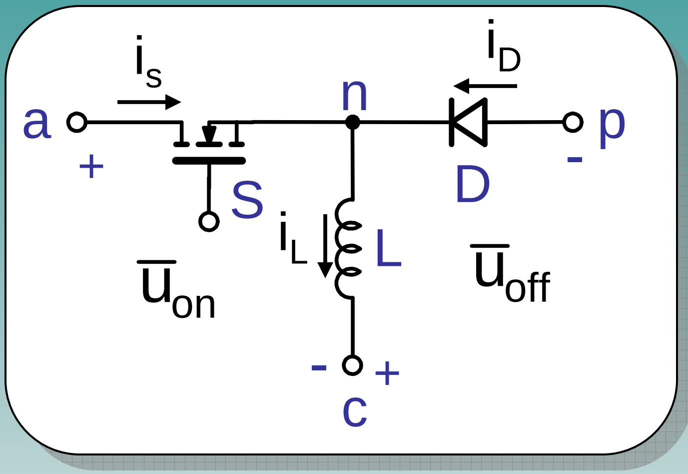
$$I_o = \bar{i}_D = \frac{d^2}{2Lf_S} U_g \frac{1}{M} \Rightarrow M = \frac{d^2}{I_{oN}}$$

First order average models - DCM

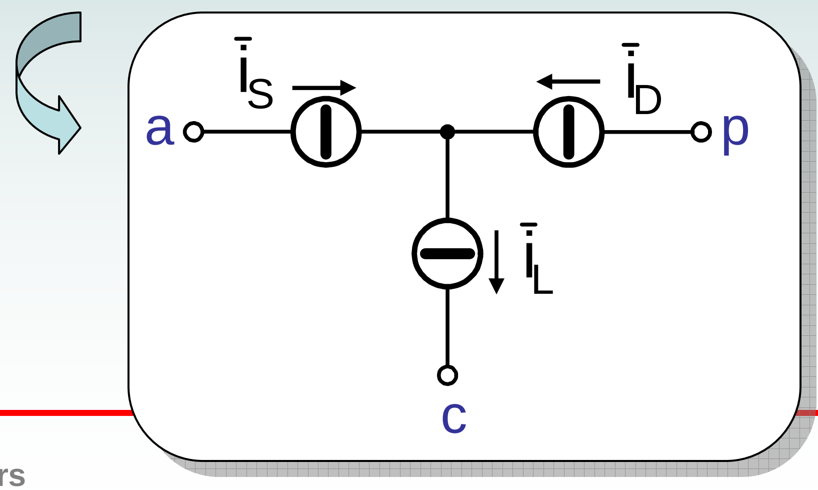
- The inductor current is always zero at the beginning of each switching period;
- this loss of the memory effect justifies the statement that the inductor current is no more a state variable;
- switch and diode are replaced by non linear controlled current generators

First order average models - DCM

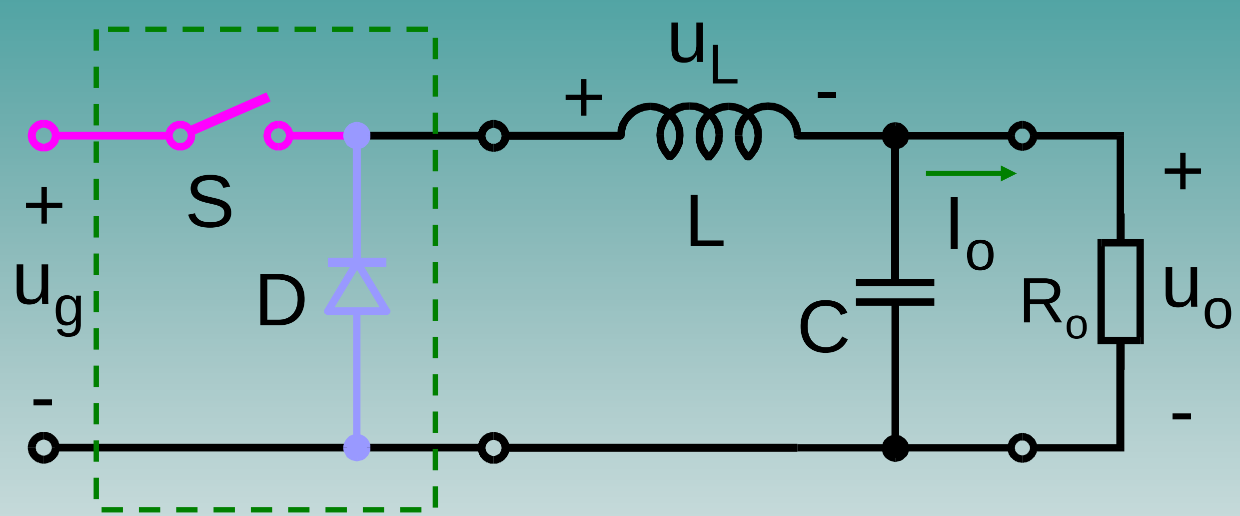


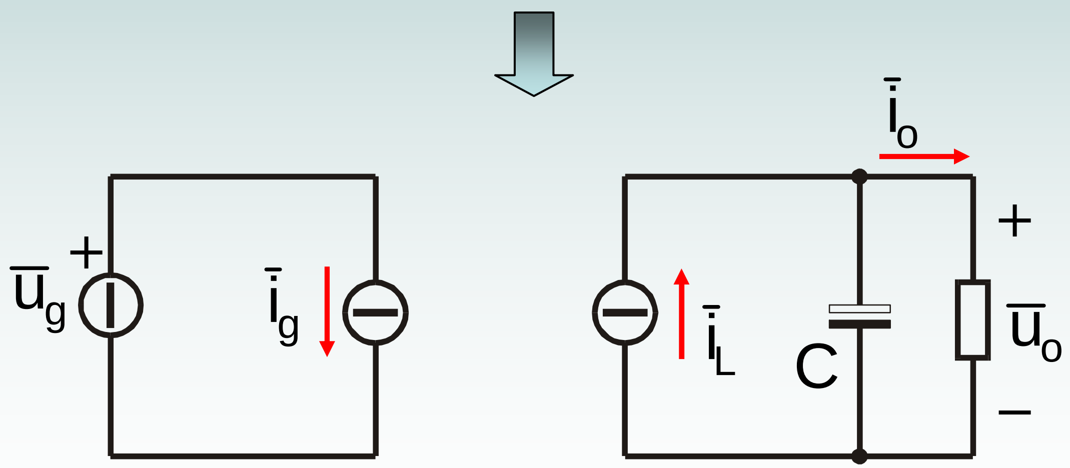


Inductor average voltage is always zero in a switching period! (?)



Buck average model: DCM





Buck small-signal model: DCM

Average quantities:

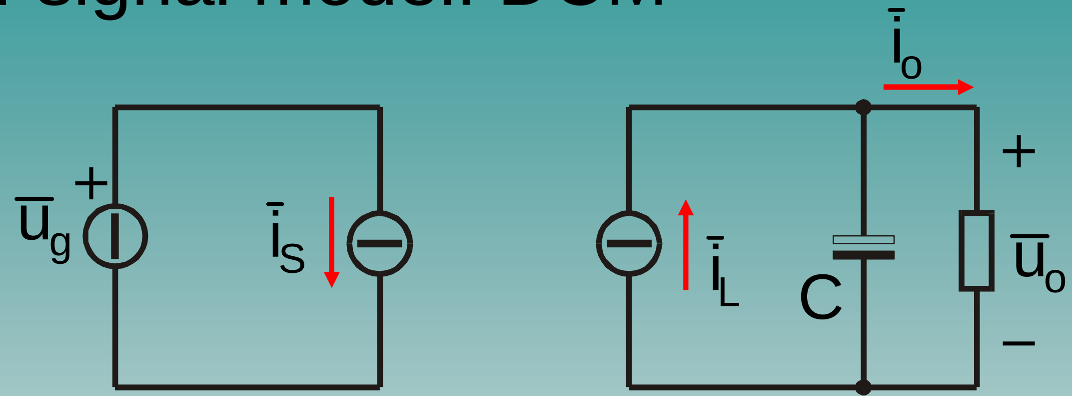
$$\begin{cases} \bar{i}_L = \dfrac{d^2}{2Lf_S}\left(\bar{u}_g - \bar{u}_o\right)\left(\dfrac{\bar{u}_g}{\bar{u}_o}\right) = h\left(\bar{u}_g, \bar{u}_o, d\right) \\ \bar{i}_g = \bar{i}_S = \dfrac{d^2}{2Lf_S}\left(\bar{u}_g - \bar{u}_o\right) = f\left(\bar{u}_g, \bar{u}_o, d\right) \end{cases}$$

⇓

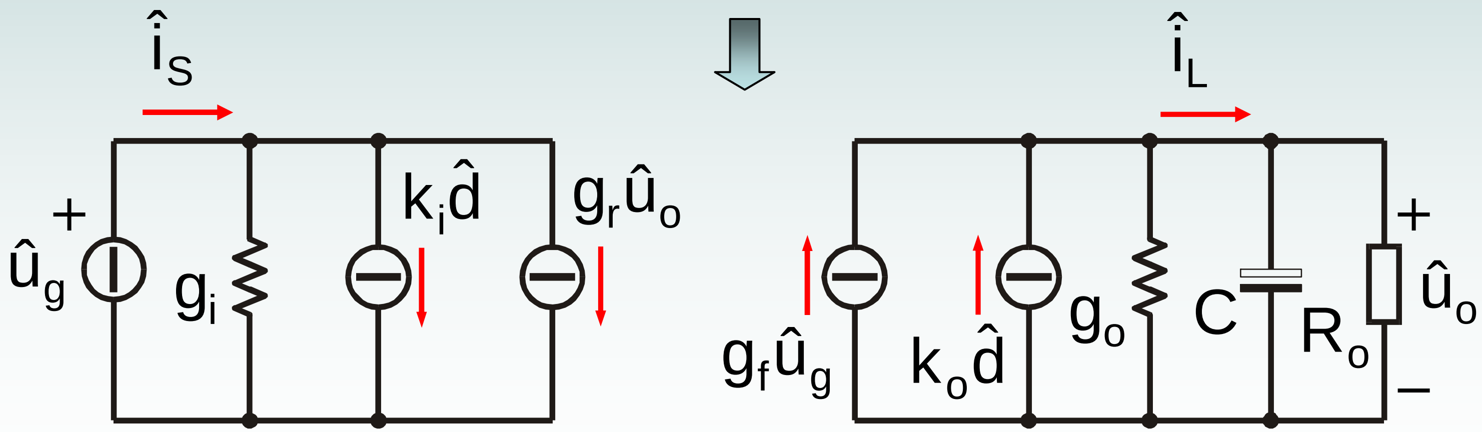
Perturbation:

$$\begin{cases} \hat{i}_g = \dfrac{\partial f}{\partial \bar{u}_g}\hat{u}_g + \dfrac{\partial f}{\partial \bar{u}_o}\hat{u}_o + \dfrac{\partial f}{\partial d}\hat{d} = g_i\hat{u}_g + g_r\hat{u}_o + k_i\hat{d} \\ \hat{i}_L = \dfrac{\partial h}{\partial \bar{u}_g}\hat{u}_g + \dfrac{\partial h}{\partial \bar{u}_o}\hat{u}_o + \dfrac{\partial h}{\partial d}\hat{d} = g_f\hat{u}_g - g_o\hat{u}_o + k_o\hat{d} \end{cases}$$

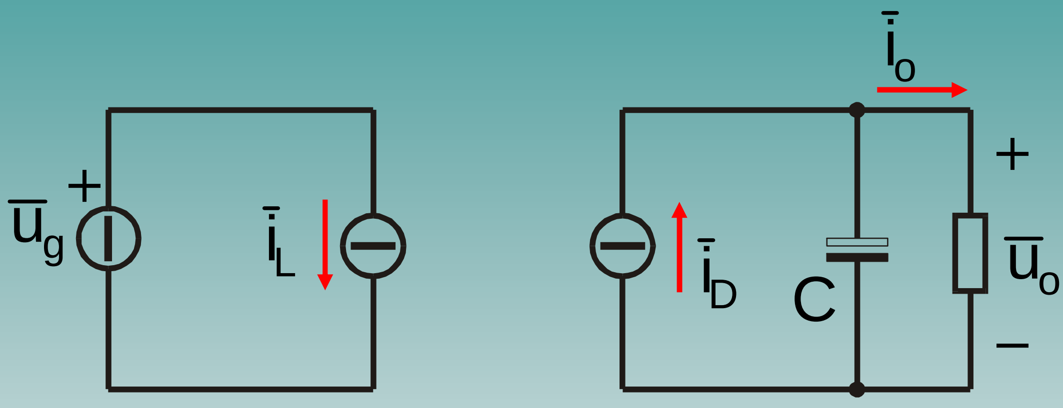
Buck small-signal model: DCM



First order model

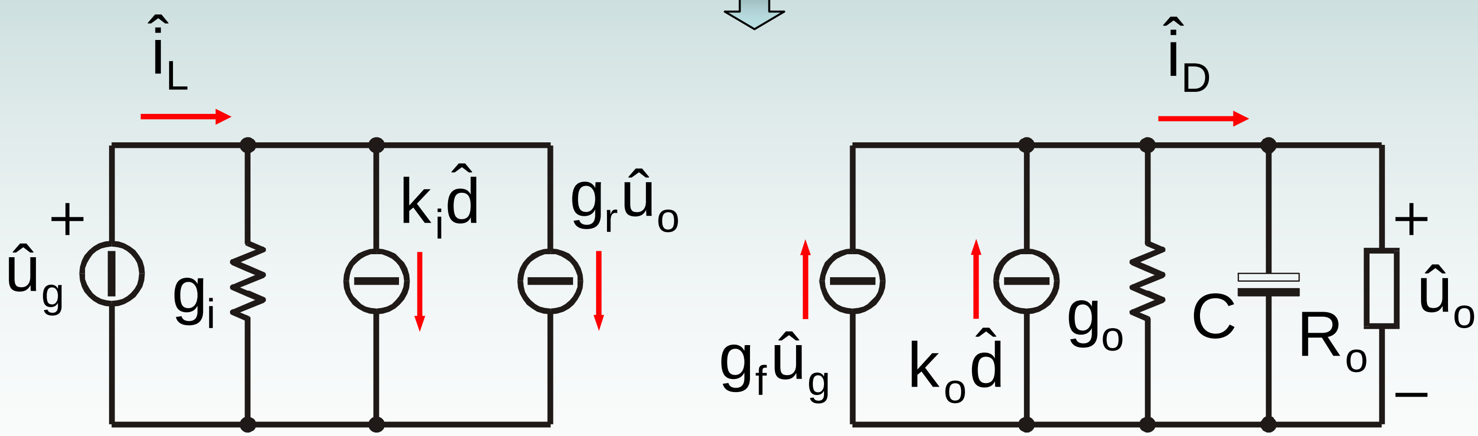


Boost small-signal model: DCM

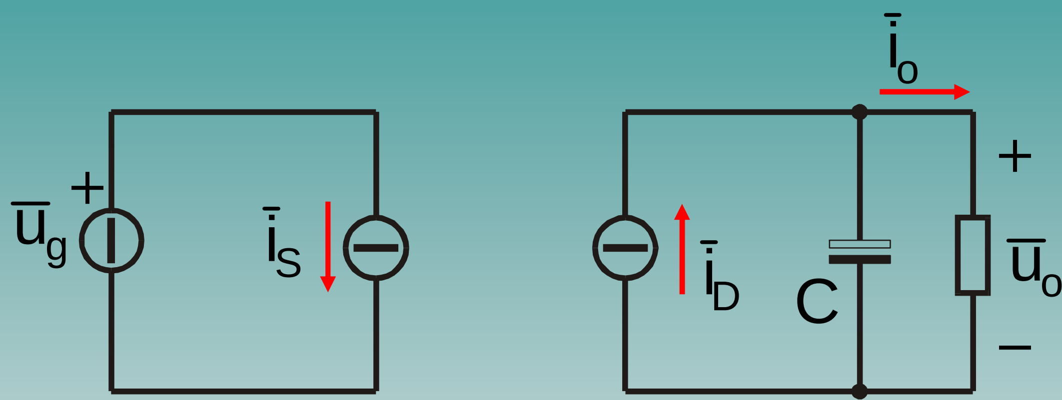


$$\bar{i}_L = \frac{d^2}{2Lf_S}\bar{u}_g\left(\frac{\bar{u}_o}{\bar{u}_o-\bar{u}_g}\right)$$

$$\bar{i}_D = \frac{d^2}{2Lf_S}\frac{\bar{u}_g^2}{\bar{u}_o-\bar{u}_g}$$

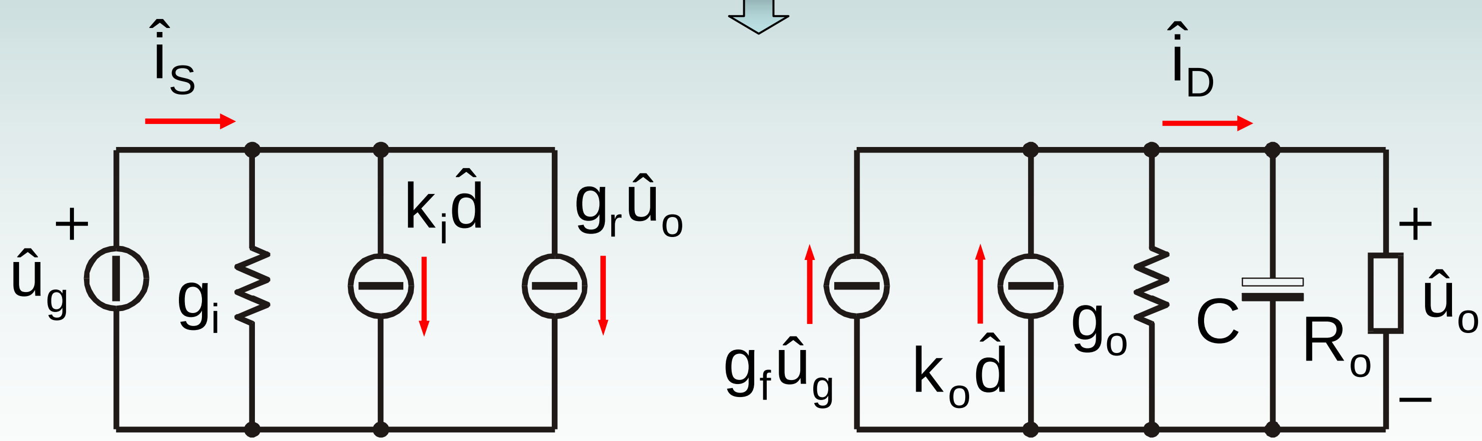


Buck-Boost small-signal model: DCM



$$\bar{i}_D = \frac{d^2}{2Lf_S}\frac{\bar{u}_g^2}{\bar{u}_o}$$

$$\bar{i}_S = \frac{d^2}{2Lf_S}\bar{u}_g$$



Full order average models: DCM

Impulsive perturbation:

$$\hat{D}(s) = \mathcal{L}\{\hat{t}_S\} = \int_0^{+\infty} \hat{t}_S(t) e^{-st} dt$$

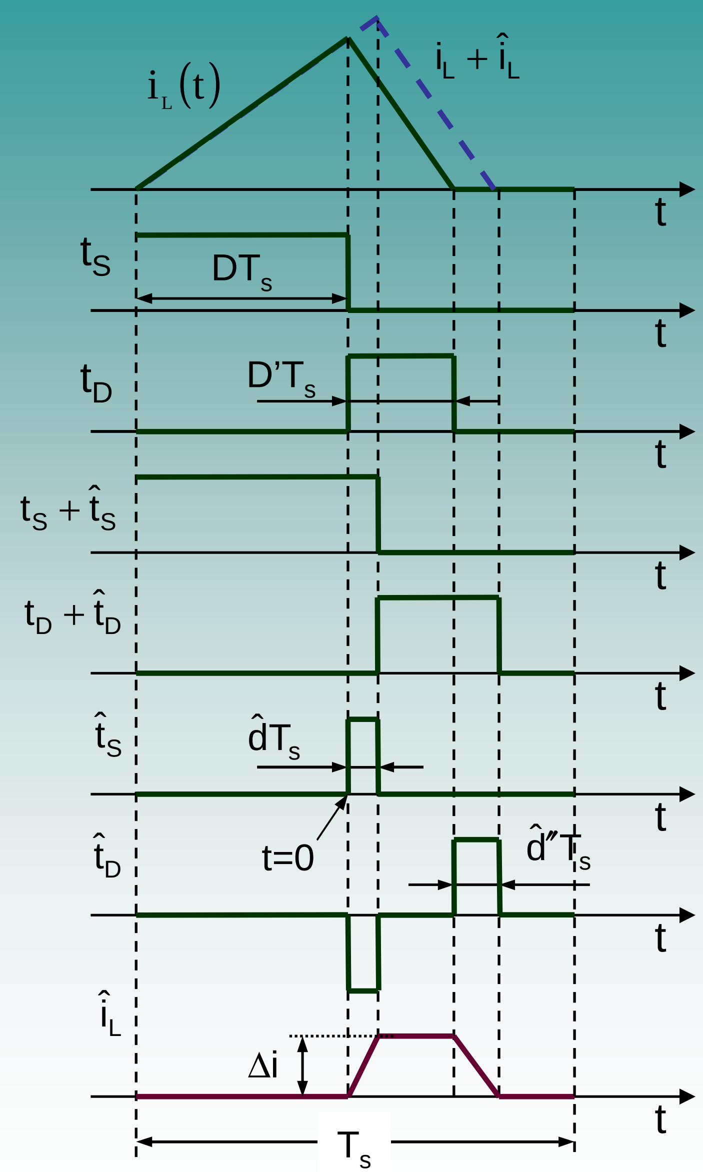
$$= \int_0^{\hat{d}T_s} e^{-st} dt = \frac{1 - e^{-s\hat{d}T_s}}{s} \approx \hat{d}T_s$$

Response to impulsive perturbation:

$$\hat{i}_L(s) = \mathcal{L}\{\hat{i}_L\} = \int_0^{+\infty} \hat{i}_L(t) e^{-st} dt = \Delta i \int_0^{D'T_s} e^{-st} dt$$

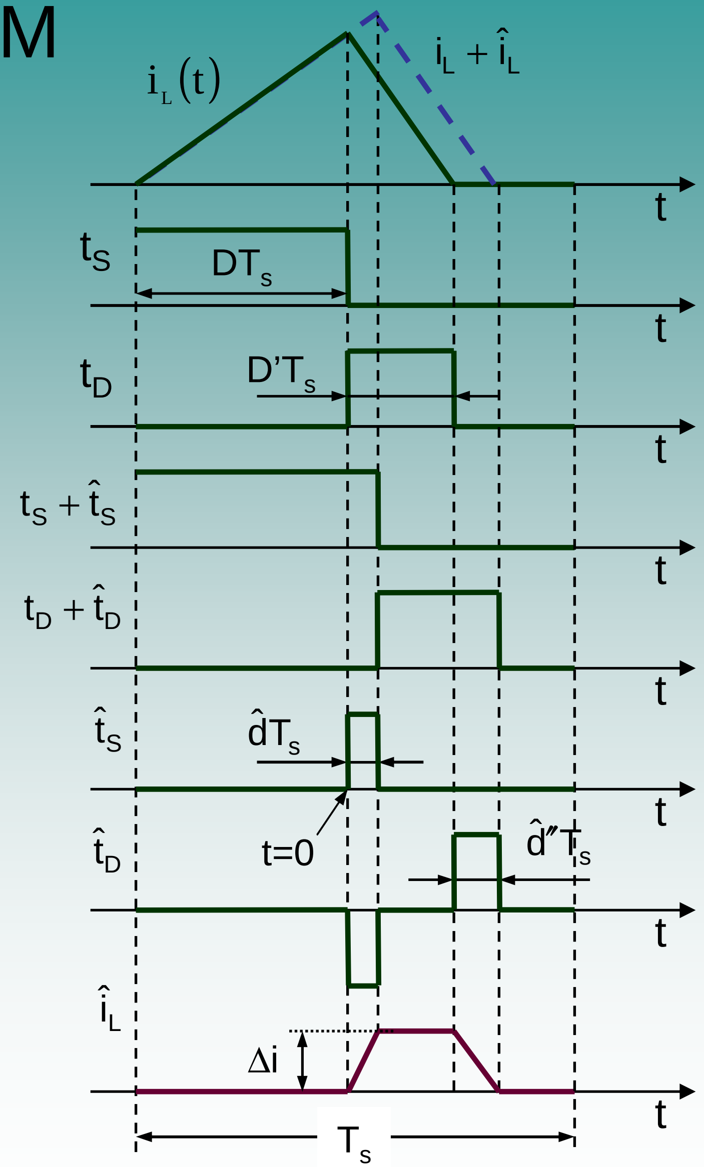
$$= \frac{\overline{u}_{on} + \overline{u}_{off}}{L} \hat{d}T_s \frac{1 - e^{-sD'T_s}}{s}$$

$$G_{id}(s) = \frac{\hat{i}_L(s)}{\hat{D}(s)} = \frac{(\overline{u}_{on} + \overline{u}_{off})D'}{Lf_s} \left(\frac{1 - e^{-sD'T_s}}{sD'T_s} \right)$$



Full order average models: DCM

$$G_{id}(s)=\frac{\hat{i}_L(s)}{\hat{D}(s)}=\frac{(\overline{u}_{on}+\overline{u}_{off})D'}{Lf_s}\left(\frac{1-e^{-sD'T_s}}{sD'T_s}\right)$$



First order Padé approximation:

$$e^{-sD'T_s}\approx\left(\frac{1-\frac{sD'T_s}{2}}{1+\frac{sD'T_s}{2}}\right)$$

⇓

$$G_{id}(s)\approx\frac{(\overline{u}_{on}+\overline{u}_{off})D'}{Lf_s}\left(\frac{1}{1+\frac{sD'T_s}{2}}\right)$$

$$\omega_p=\frac{2f_s}{D'}\quad\Rightarrow\quad f_p=\frac{f_s}{\pi D'}$$

Full order average models: DCM

$$\hat{d}''T_s = \frac{L\Delta i}{\overline{u}_{off}} \quad \Rightarrow \quad \hat{d}'' = \left(1 + \frac{\overline{u}_{on}}{\overline{u}_{off}}\right)\hat{d}$$

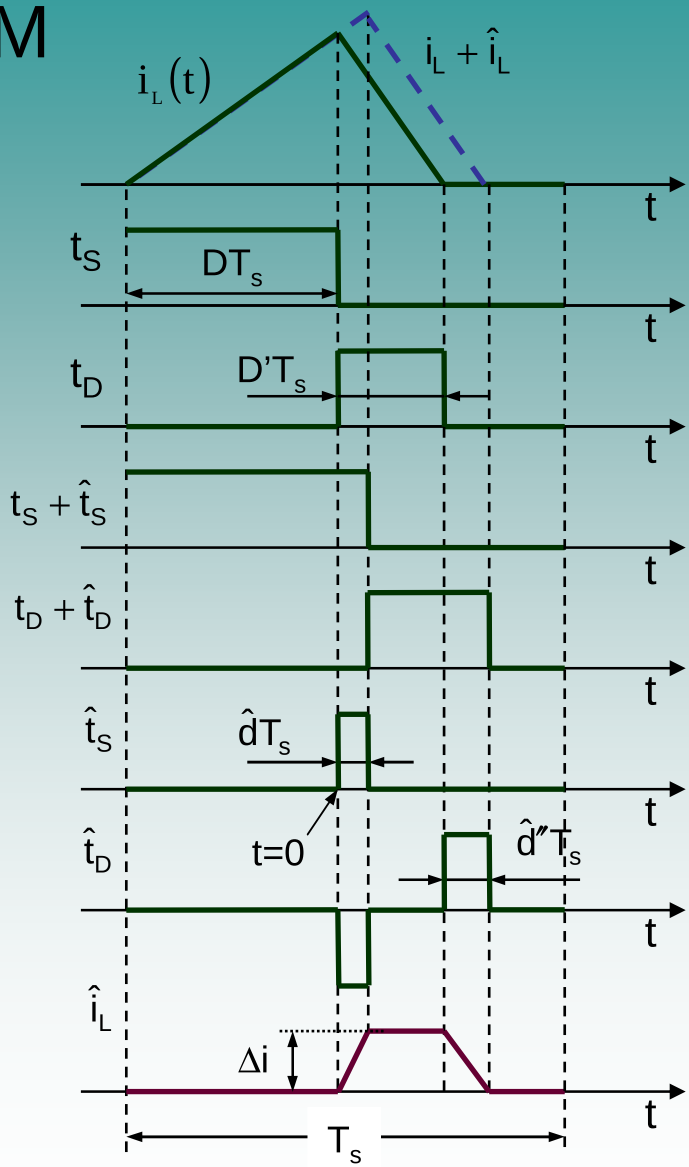
Overall d' perturbation:

$$\hat{d}' = \hat{d}'' - \hat{d} = \frac{\overline{u}_{on}}{\overline{u}_{off}}\hat{d}$$ WRONG!

$$\hat{t}_D = -\hat{d}T_s \cdot \delta(t) + \hat{d}''T_s \cdot \delta(t - D'T_s)$$

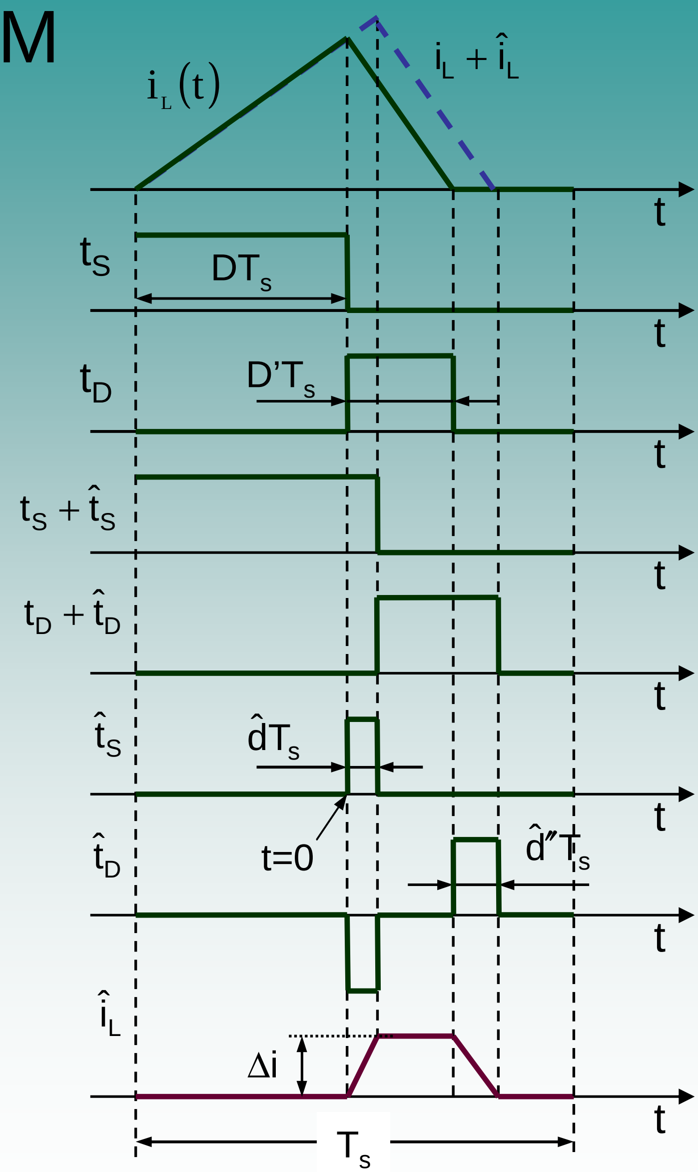
Dirac function

$$\hat{D}'(s) = \mathcal{L}\{\hat{t}_D\} = -\hat{D}(s) + \left(1 + \frac{\overline{u}_{on}}{\overline{u}_{off}}\right)\hat{D}(s) \cdot e^{-sD'T_s}$$



Full order average models: DCM

$$\frac{\hat{D}'(s)}{\hat{D}(s)} = -1 + \left(1 + \frac{\overline{u}_{on}}{\overline{u}_{off}}\right) e^{-sD'T_s}$$



Inductor current perturbation:

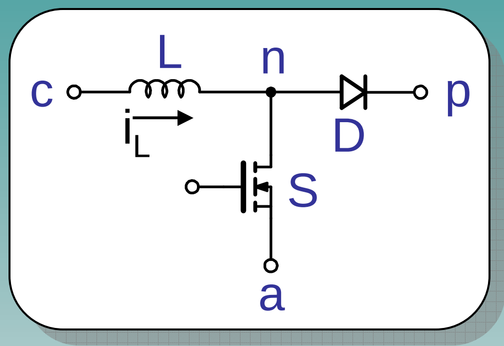
$$L\frac{d\hat{i}_L}{dt} = \hat{d}\,\overline{u}_{on} - \hat{d}'\,\overline{u}_{off}$$

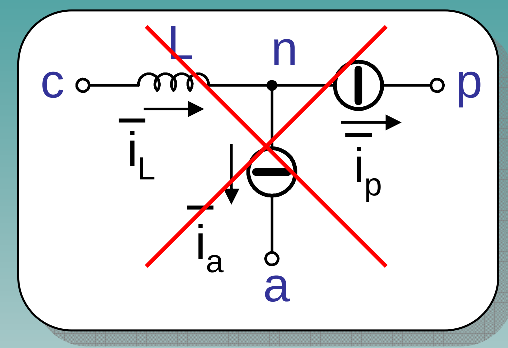
$$sL\,\hat{i}_L(s) = \hat{D}(s)\overline{u}_{on} - \hat{D}'(s)\overline{u}_{off}$$

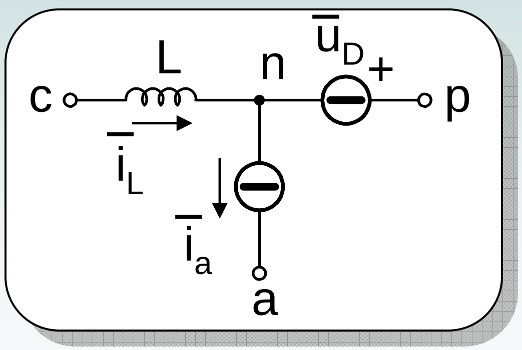
⇓

$$G_{id}(s) = \frac{\hat{i}_L(s)}{\hat{D}(s)} = \frac{(\overline{u}_{on} + \overline{u}_{off})D'}{Lf_s}\left(\frac{1 - e^{-sD'T_s}}{sD'T_s}\right)$$

Full order average models: DCM

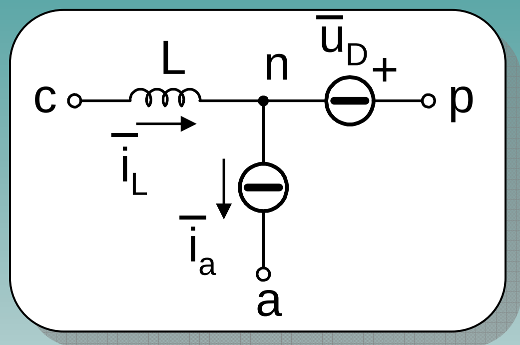


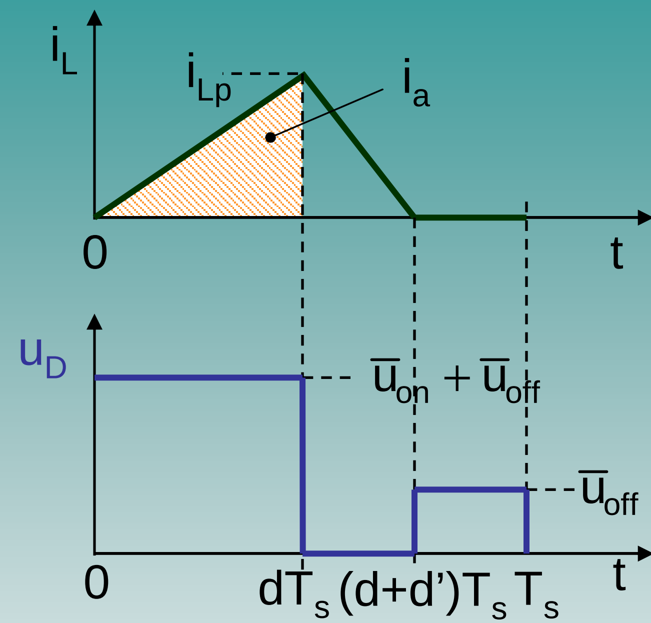




The switch is replaced by a controlled current generator while the diode is replaced by a controlled voltage generator

Full order average models: DCM





$$\begin{cases} \bar{i}_L = \frac{1}{2} i_{Lp} (d + d') \\ \bar{i}_a = \frac{1}{2} i_{Lp} d \end{cases} \Rightarrow \quad \bar{i}_a = \bar{i}_L \frac{d}{d + d'} \qquad \bar{u}_D = (\bar{u}_{on} + \bar{u}_{off}) d + \bar{u}_{off} (1 - d - d')$$

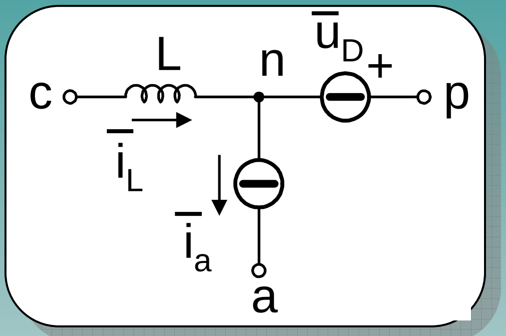
$$\bar{i}_L = \frac{\bar{u}_{on}}{2Lf_s} d(d + d') \quad \Rightarrow \quad d' = \frac{2Lf_s \bar{i}_L}{d\bar{u}_{on}} - d$$

Example: boost in DCM

$$\frac{d\bar{i}_L}{dt} = \frac{1}{L}\left[\bar{u}_g - \bar{u}_o + \bar{u}_o d + \left(\bar{u}_o - \bar{u}_g\right)\left(1 - d - d'\right)\right] =$$

$$= \left(1 - \frac{\bar{u}_o}{\bar{u}_g}\right)\frac{2f_s}{d}\bar{i}_L + \frac{\bar{u}_o d}{L}$$

$$\frac{d\bar{u}_o}{dt} = \frac{1}{C}\left(\bar{i}_L - \bar{i}_a - \frac{\bar{u}_o}{R_o}\right) = \frac{\bar{i}_L}{C} - \frac{d^2\bar{u}_g}{2LCf_s} - \frac{\bar{u}_o}{CR_o}$$



$$\bar{u}_D = \bar{u}_o d + \left(\bar{u}_o - \bar{u}_g\right)\left(1 - d - d'\right)$$

$$d' = \frac{2Lf_s\bar{i}_L}{d\bar{u}_g} - d$$

Duty-cycle perturbation:

$$\begin{cases} \dfrac{d\hat{i}_L}{dt} = \left(1 - \dfrac{U_o + \hat{u}_o}{U_g}\right)\dfrac{2f_s}{D + \hat{d}}\left(I_L + \hat{i}_L\right) + \dfrac{\left(U_o + \hat{u}_o\right)\left(D + \hat{d}\right)}{L} \\ \dfrac{d\hat{u}_o}{dt} = \dfrac{I_L + \hat{i}_L}{C} - \dfrac{\left(D + \hat{d}\right)^2 U_g}{2LCf_s} - \dfrac{U_o + \hat{u}_o}{CR_o} \end{cases}$$

Example: boost in DCM

Small-signal linear model:

$$\begin{cases} \dfrac{d\hat{i}_L}{dt} = \dfrac{2f}{D}(1-M)\hat{i}_L + \dfrac{2U_o}{L}\hat{d} - \dfrac{2Mf_s}{DR_o}\hat{u}_o \\ \dfrac{d\hat{u}_o}{dt} = \dfrac{\hat{i}_L}{C} - \dfrac{DU_g}{LCf_s}\hat{d} - \dfrac{\hat{u}_o}{CR_o} \end{cases}$$

$$G_{ud}(s) = \frac{\hat{U}_o(s)}{\hat{D}(s)} = \frac{DU_g}{LCf_s}\frac{\left(\dfrac{2f_s}{D} - s\right)}{s^2 + s\left(\dfrac{1}{R_oC} + \dfrac{2f_s(M-1)}{D}\right) + \dfrac{2f_s(2M-1)}{DR_oC}} \approx K_B \frac{\left(1 - \dfrac{s}{\omega_z}\right)}{\left(1 + \dfrac{s}{\omega_{pBF}}\right)\left(1 + \dfrac{s}{\omega_{pAF}}\right)}$$

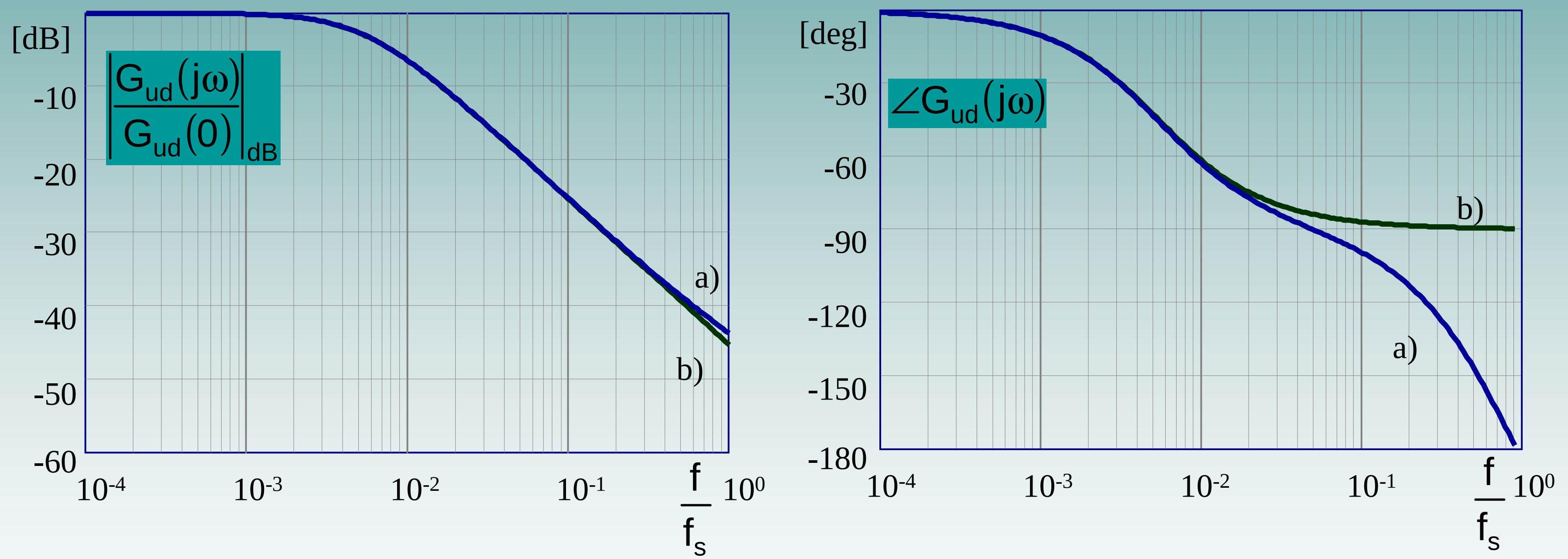
$$K_B = \frac{2U_g}{2M-1}\sqrt{\frac{M(M-1)}{k}} \qquad \omega_z = \frac{2f_s}{D} \qquad \omega_{pLF} = \frac{2f_s(M-1)}{D}$$

The same as first order model:

$$\omega_{pHF} = \frac{1}{R_oC}\left(\frac{2M-1}{M-1}\right)$$

Example: boost in DCM

Control-to-output transfer function



a) Full order model
b) First order model

State-Space averaging (SSA): CCM

State, input and output variable vector:

$$x = \begin{bmatrix} i_L \\ u_C \end{bmatrix} \quad u = \begin{bmatrix} u_g \\ i_o \end{bmatrix} \quad y = \begin{bmatrix} u_o \\ i_g \end{bmatrix}$$

Interval dT_s:

$$\begin{cases} \dot{x} = A_1 x + B_1 u \\ y = C_1 x \end{cases}$$

Interval $(1\text{-}d)T_s$:

$$\begin{cases} \dot{x} = A_2 x + B_2 u \\ y = C_2 x \end{cases}$$

Switching function: $q(t) = \begin{cases} 1 & t \in t_{on} \\ 0 & t \in t_{off} \end{cases}$

$$\begin{cases} \dot{x} = A_2 x + B_2 u + [(A_1 - A_2)x + (B_1 - B_2)u] \cdot q = A_2 x + B_2 u + F \cdot q \\ y = C_2 x + (C_1 - C_2)x \cdot q = C_2 x + G \cdot q \end{cases}$$

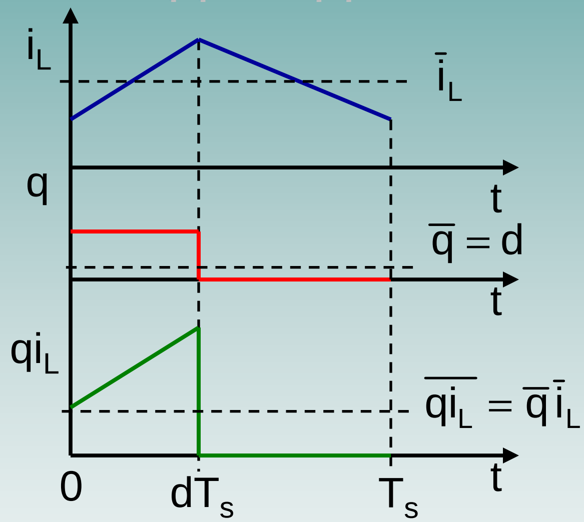
Applying moving average operator: $\overline{\dot{x}} = \dot{\overline{x}}$ ⇨

$$\begin{cases} \dot{\overline{x}} = A_2 \overline{x} + B_2 \overline{u} + \overline{F \cdot q} \\ \overline{y} = C_2 \overline{x} + \overline{G \cdot q} \end{cases}$$

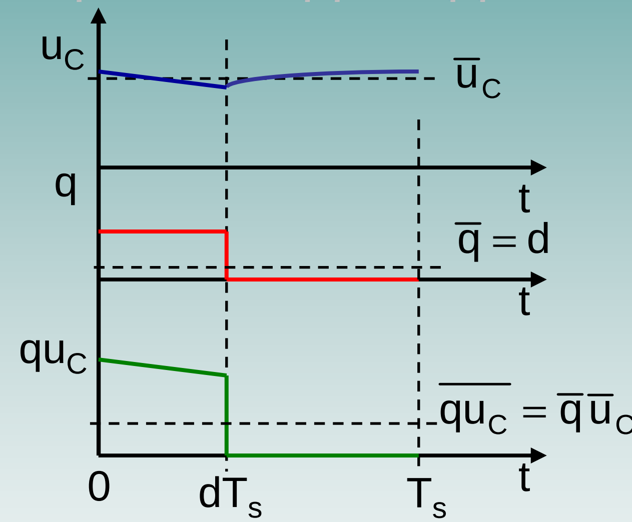
State-Space averaging (SSA): CCM

$$\overline{F \cdot q} = ?, \quad \overline{G \cdot q} = ?$$

Hp: linear ripple approximation



Hp: small ripple approximation



$$\overline{F \cdot q} = \overline{F} \cdot \overline{q} = \overline{F} \cdot d$$

$$\overline{G \cdot q} = \overline{G} \cdot \overline{q} = \overline{G} \cdot d$$

$$\Rightarrow \begin{cases} \dot{\overline{x}} = A_2 \overline{x} + B_2 \overline{u} + \overline{F} d = A\overline{x} + B\overline{u} \\ \overline{y} = C_2 \overline{x} + \overline{G} d = C\overline{x} \end{cases}$$

$$A = A_1 d + A_2 (1-d) \quad B = B_1 d + B_2 (1-d) \quad C = C_1 d + C_2 (1-d)$$

State-Space averaging (SSA): CCM

Steady-state solution:

$$\begin{cases} 0 = A\overline{x} + B\overline{u} \\ \overline{y} = C\overline{x} \end{cases} \Rightarrow \begin{cases} \overline{x} = -A^{-1}B\overline{u} \\ \overline{y} = C\overline{x} \end{cases}$$

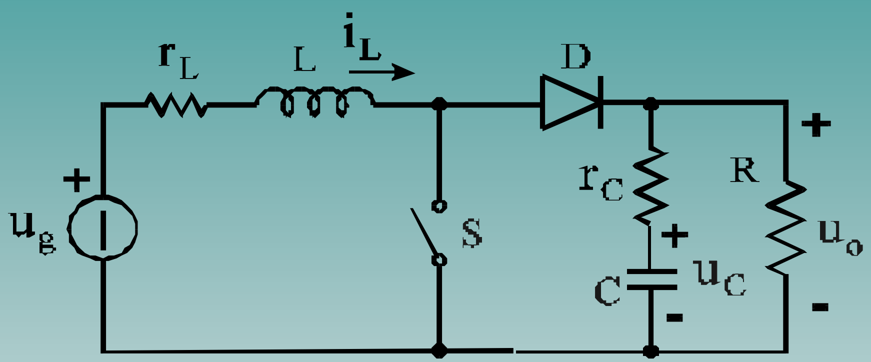
Small-signal linear model:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + B\hat{u} + \left[(A_1 - A_2)X + (B_1 - B_2)U\right]\hat{d} = A\hat{x} + B\hat{u} + F\hat{d} \\ \hat{y} = C\hat{x} + (C_1 - C_2)X\ \hat{d} = C\hat{x} + G\hat{d} \end{cases}$$

⇓

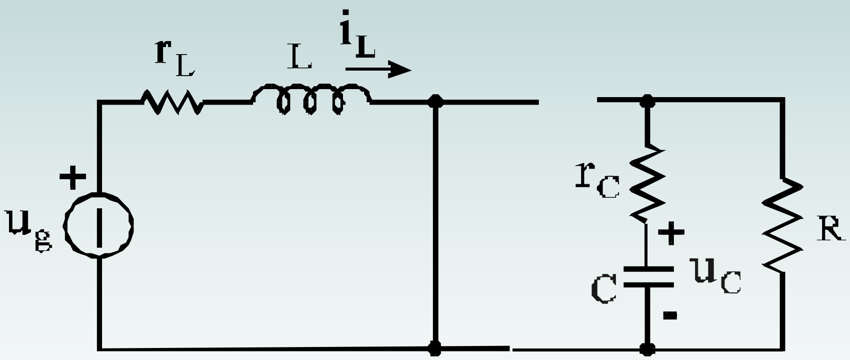
$$\begin{cases} \hat{X}(s) = (sI - A)^{-1}\left(B\hat{U}(s) + F\hat{D}(s)\right) \\ \hat{Y}(s) = C\hat{X}(s) + G\hat{D}(s) \end{cases}$$

Example: boost converter in CCM



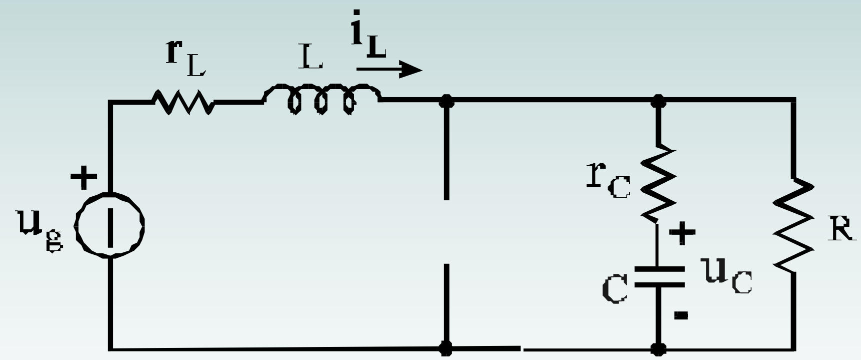
$$x = \begin{bmatrix} i_L \\ u_C \end{bmatrix} \qquad u = u_g \qquad y = u_o$$

Interval dT_s:



$$\begin{cases} \dot{x} = A_1 x + B_1 u \\ y = C_1 x \end{cases}$$

Interval $(1\text{-}d)T_s$:



$$\begin{cases} \dot{x} = A_2 x + B_2 u \\ y = C_2 x \end{cases}$$

Example: boost converter in CCM

$$A_1 = \begin{bmatrix} -\dfrac{r_L}{L} & 0 \\ 0 & -\dfrac{1}{(R+r_C)C} \end{bmatrix} \qquad A_2 = \begin{bmatrix} -\dfrac{r_L + r_C // R}{L} & -\dfrac{R}{(R+r_C)L} \\ \dfrac{R}{(R+r_C)C} & -\dfrac{1}{(R+r_C)C} \end{bmatrix} \qquad B_1 = B_2 = \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & \dfrac{R}{R+r_C} \end{bmatrix} \qquad C_2 = \begin{bmatrix} R // r_C & \dfrac{R}{R+r_C} \end{bmatrix}$$

Steady-state solution:

$$\dot{x} = 0 \quad \Rightarrow \quad x = -A^{-1} B U_g$$

$$\begin{cases} X = \begin{bmatrix} I_L \\ U_C \end{bmatrix} = \dfrac{U_g}{R'} \begin{bmatrix} 1 \\ D'R \end{bmatrix} \\ Y = U_o = \dfrac{U_g D' R}{R'} \end{cases}$$

$$R' = D'^2 R + r_L + DD'(r_C // R)$$

Voltage conversion ratio:

$$M = \frac{U_o}{U_g} = \frac{1}{D'} \cdot \frac{D'^2 R}{D'^2 R + r_L + DD'(r_C // R)}$$

Example: boost converter in CCM

Small-signal linear model:

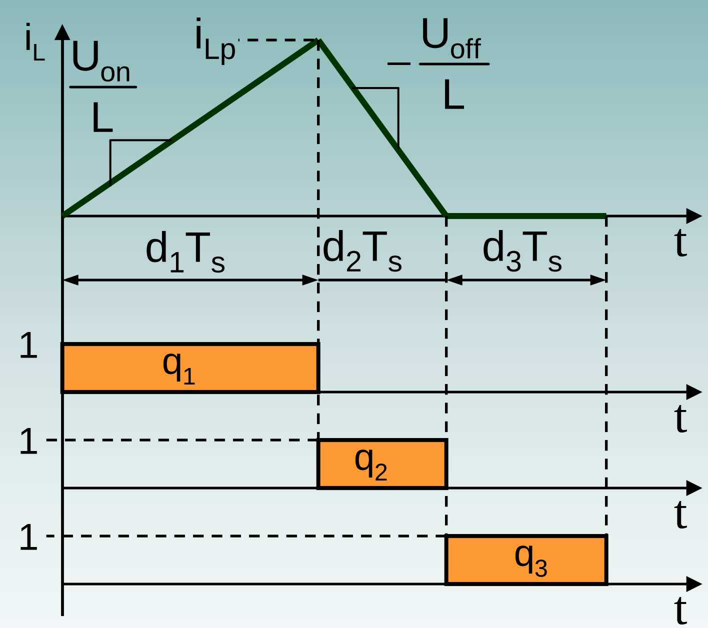
$$\begin{bmatrix} \dot{\hat{i}}_L \\ \dot{\hat{u}}_C \end{bmatrix} = \begin{bmatrix} -\dfrac{r_L + D'(r_C // R)}{L} & -\dfrac{D'R}{(R+r_C)L} \\ \dfrac{D'R}{(R+r_C)C} & -\dfrac{1}{(R+r_C)C} \end{bmatrix} \begin{bmatrix} \hat{i}_L \\ \hat{u}_C \end{bmatrix} + \begin{bmatrix} \dfrac{1}{L} \\ 0 \end{bmatrix} \hat{u}_g + \frac{U_g}{R'} \begin{bmatrix} \dfrac{R(D'R + r_C)}{(R+r_C)L} \\ -\dfrac{R}{(R+r_C)C} \end{bmatrix} \hat{d}$$

$$\hat{y} = \begin{bmatrix} D'(r_C \| R) & \dfrac{R}{R+r_C} \end{bmatrix} \begin{bmatrix} \hat{i}_L \\ \hat{u}_C \end{bmatrix} - U_g \frac{r_C \| R}{R'} \hat{d}$$

State-Space averaging (SSA): DCM

State variable vector: $x = \begin{bmatrix} i_L \\ u_C \end{bmatrix}$

$$\dot{x}(t) = \left(\sum_{k=1}^{3} q_k A_k \right) x + \left(\sum_{k=1}^{3} q_k B_k \right) u$$



Applying moving average operator: $\bar{\dot{x}} = \dot{\bar{x}}$

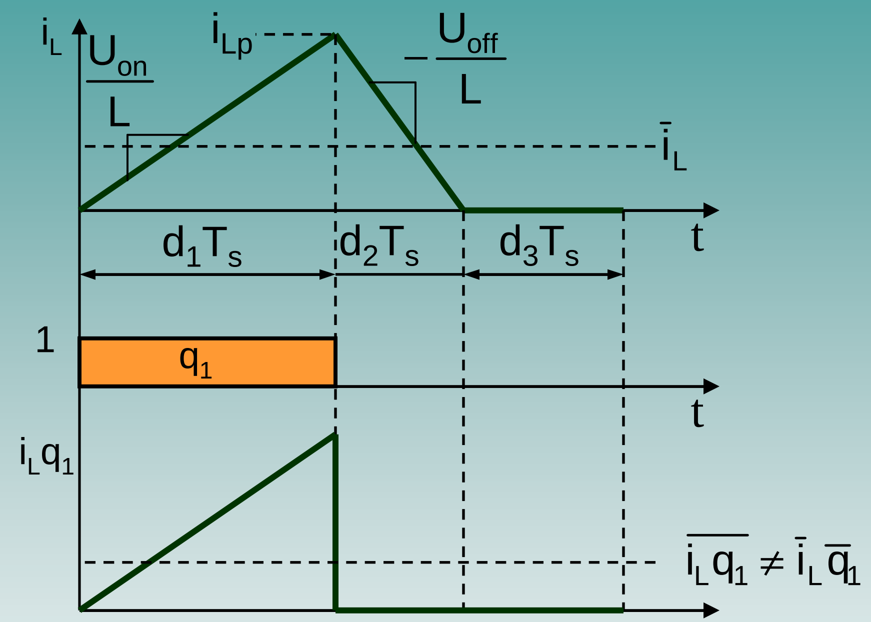
$$\dot{\bar{x}} \neq \left(\sum_{k=1}^{3} d_k A_k \right) \bar{x} + \left(\sum_{k=1}^{3} d_k B_k \right) \bar{u}$$

Why?

State-Space averaging (SSA): DCM

Hp: linear ripple approximation

Example: $i_L \cdot q_1$



$$\bar{i}_L = \frac{i_{Lpk}}{2}(d_1 + d_2)$$

$$\overline{q_2 i_L} = \bar{i}_D = \frac{i_{Lpk}}{2} d_2 = \bar{i}_L \frac{d_2}{d_1 + d_2}$$

$$= \bar{i}_L \bar{q}_2 \frac{1}{d_1 + d_2}$$

Corrective term

$$\overline{q_1 i_L} = \bar{i}_S = \frac{i_{Lpk}}{2} d_1 = \bar{i}_L \frac{d_1}{d_1 + d_2} = \bar{i}_L \bar{q}_1 \frac{1}{d_1 + d_2}$$

State-Space averaging (SSA): DCM

Hp: small ripple approximation

$$\overline{q_i u_C} = \overline{q_i}\,\overline{u_C} = d_i \overline{u}_C \qquad i = 1,2,3$$

$$\dot{x}(t) = \left(\sum_{k=1}^{3} q_k A_k\right) x + \left(\sum_{k=1}^{3} q_k B_k\right) u \quad \Rightarrow \quad \dot{\overline{x}} = \left(\sum_{k=1}^{3} d_k A_k\right) M \overline{x} + \left(\sum_{k=1}^{3} d_k B_k\right) \overline{u}$$

M is the correction matrix

$$M = \begin{bmatrix} \dfrac{1}{d_1 + d_2} & 0 \\ 0 & 1 \end{bmatrix}$$

State-Space averaging (SSA): DCM

$$\dot{\overline{x}} = \left(\sum_{k=1}^{3} d_k A_k \right) M \overline{x} + \left(\sum_{k=1}^{3} d_k B_k \right) \overline{u}$$

$$M = \begin{bmatrix} \frac{1}{d_1 + d_2} & 0 \\ 0 & 1 \end{bmatrix} \qquad d_2 = \frac{2 L f_s \overline{i}_L}{\delta \overline{u}_{on}} - d_1 \qquad d_3 = 1 - d_1 - d_2$$

If $d_3 = 0$, i.e. in CCM, we have:

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \dot{\overline{x}} = \left(\sum_{k=1}^{2} d_k A_k \right) \overline{x} + \left(\sum_{k=1}^{2} d_k B_k \right) \overline{u}$$